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CARDINAL INTERPOLATION

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THE OHIO STATE UNIVERSITY
RESEARCH FOUNDATION
1958 Neil Avenue
Columbus, Ohio 43210

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Base functions of various kind can be used for prediction problems. All of them are interrelated to each other. In particular, the family of splines, the Gaussian function, the $\sin\pi x/\pi x$ function, the Hirvonen covariance functions, and the Bjerhammar interpolation functions are considered here. Particular emphasis is put on the important role of the correlation length of the base (covariance) function; its strong relations to the data sampling

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rate and to the maximum prediction error should advantageously be used as a guideline for the design of gravity field data-collection and data-processing programs.

FOREWORD

This report was prepared by Dr. Hans Sünkel, Technical University at Graz, Austria, under Air Force Contract No. F19628-79-C-0075, The Ohio State University Research Foundation, Project No. 711715, Project Supervisor, Urho A. Uotila, Professor, Department of Geodetic Science. The contract covering this research is administered by the Air Force Geophysics Laboratory (AFGL), Hanscom Air Force Base, Massachusetts, with Bela Szabo, Project Scientist.

CONTENTS

INTRODUCTION

Essential features of any mathematical method can be studied best if extreme cases are considered. As far as interpolation, filtering, prediction, or even least-squares collocation are concerned, an infinite homogeneous set of regularly distributed data presents itself as an ideal candidate for such kind of studies. Without restriction of generality one can assume a unit distance between neighboring data points, such that the data are located at the places of the cardinal numbers on the real line; this is why we have used the term "cardinal" in the title. The second term "interpolation" does not require a further interpretation, although the present study goes beyond that.

The goal is to investigate the response of the choice of various base (covariance) functions onto the interpolated (predicted) function based on an infinite homogeneous data set.

Chapter 1 deals with the whole family of spline base functions, starting with the zero degree spline base function (= step function) and finally arriving at the highest degree spline, whose corresponding sampling function equals the $\sin\pi x/\pi x$ function.

In the second chapter, relations between Gaussian functions of various correlation lengths and the corresponding spline functions of certain degree are established. Deviations of these two functions from each other are estimated.

Structural similarities between spline interpolation and least-squares interpolation are discussed in chapter 3. The finite support of splines and their ability to fit and replace any kind of covariance (= base) function leads to the conclusion that splines of high and odd degree may advantageously be used as base functions for the solution of least-squares prediction problems.

A virtually completely different interpolation method, which has been advocated by Bjerhammar and others, is investigated in chapter 4 . An interesting and remarkable relation to spline functions of odd degree is presented.

The impact of data noise and correlation length onto the sampling functions and the prediction error is investigated in chapter 5 . Of particular importance, for all considerations concerning data collection, is a very strong relation between the correlation length of the underlying covariance function, the sampling rate (distance between data points), the data noise, and the maximum prediction error. Its impact on practical gravity field sampling problems might be considerable.

There exists a whole bunch of base functions which can be used for interpolation and prediction problems. It is astonishing and remarkable that such strong ties exist between all of them. The beautiful interplay between the physical and the frequency domain is given a dominant role in all investigations performed here.

1. SPLINE INTERPOLATION ON THE REAL LINE

Let us give a definition of the key words appearing in the above title. There exist basically two different types of interpolation functions: the ones are based on the intrinsic statistical information contained in a data set, the others are not; the latter ones are referred to as "deterministic" interpolation functions. By the term "local" we understand an interpolant which is sensitive with respect to data in the neighborhood of the interpolation point and blind or considerably less sensitive with respect to remote data. (A typical counter-example would be a single interpolating polynomial.) An interpolant is understood as a function which reproduces the data. When talking about "data" we have a homogeneous set of error-free data in mind which is regularly distributed along a line; without restriction of generality we can assume the data to be located at the cardinal numbers $\ldots, -2, -1, 0, 1, 2, \ldots$; the data set is supposed to have infinite dimension, it is denoted by the infinite vector pair

$$\underline{x} = \{j\} , \quad \underline{f} = \{f_j\} , \quad j = \ldots, -2, -1, 0, 1, 2, \ldots ,$$

with $\underline{x}$ denoting the vector of coordinates and $\underline{f}$ the vector of corresponding function values.

In the sequel interpolants will be studied which differ by their interpolation and differentiation properties, or simply, by their smoothness.

1.1 THE STEP FUNCTION "INTERPOLATION"

In many numerical integration problems the most primitive functional representation of the integrand is chosen, essentially because of simplicity (e.g. Stokes' integral); it is the step-function representation of the "true" function which is here sampled at the cardinal numbers. The base function is the discontinuous and strictly local function (Fig. 1.1)

$$\phi_0(x) = \begin{cases} 1 & \text{for} \quad |x| < \frac{1}{2} \\ \frac{1}{2} & \text{for} \quad |x| = \frac{1}{2} \\ 0 & \text{else} \end{cases} \qquad (1.1)$$

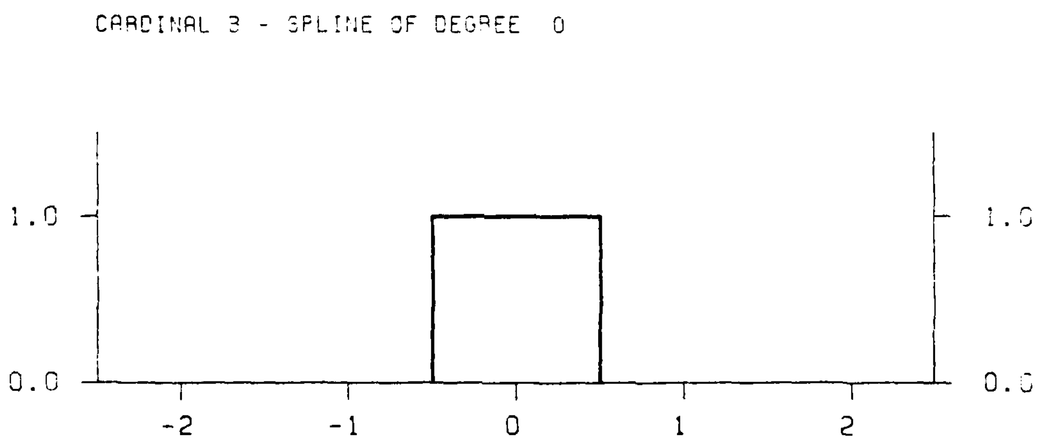

FIG. 1.1 Base function for K^0

It is an element out of K^0 , the space of discontinuous but quadratically integrable functions. The corresponding "interpolant" is simply given by the linear combination

$$\hat{f}(x) = \sum_{-\infty}^{\infty} f_m \phi_0^{(m)}(x) \quad . \tag{1.2}$$

Note that the coefficients of the linear combination are simply the function values at the grid points. In this context $\hat{f}$ denotes the interpolating function, which is an approximation to the original (and unknown) function f, $\hat{f} \approx f$; $\phi_0^{(m)}(x)$ is the base function corresponding to the point $x = m$. Observing that

$$\phi_0^{(m)}(x) = \phi_0(x - m) \tag{1.3a}$$

and

$$f_m = f(m) \quad , \tag{1.3b}$$

equation (1.2) can alternatively be written as

$$\hat{f}(x) = \sum_{-\infty}^{\infty} f(m)\phi_0(x - m) \quad , \tag{1.4}$$

which represents a product of the infinite vector f and the vector ϕ_0 whose elements are space-shifted base functions.

1.1.1 The spectrum of the step function

Everybody knows from school algebra that mathematical operations can be considerably simplified if performed in the "logarithmic domain". Multiplications and divisions are transformed

to additions and subtractions, etc.; the logarithmic function $\ln(\cdot)$ transforms from the "space domain" into the "logarithmic domain", the exponential function $\exp(\cdot)$ transforms in the counter-direction. Therefore, $\ln(\cdot)$ and $\exp(\cdot)$ are termed "transform pairs".

As far as functions are concerned, a particularly useful tool is provided by the Fourier transform. If the space domain function $f(x)$ is absolutely integrable,

$$\int_{-\infty}^{\infty} |f(x)| dx < \infty \ , \tag{1.5}$$

then the Fourier transform and its inverse transform exist,

$$F(\eta) = \int_{-\infty}^{\infty} f(x) e^{-i2\pi\eta x} dx \tag{1.6a}$$

$$f(x) = \int_{-\infty}^{\infty} F(\eta) e^{i2\pi\eta x} d\eta \ . \tag{1.6b}$$

Note that the Fourier transform $F(\eta)$ is, in general, a complex function. $f(x)$ and $F(\eta)$ are frequently called "Fourier transform pairs". $f(x)$ is the space function, $F(\eta)$ the corresponding frequency function, η the frequency.

Let us consider the Fourier transfrom of the step-base function $\phi(x)$ as defined by equation (1.1)

$$\Phi_0(\eta) = \int_{-\infty}^{\infty} \phi_0(x) e^{-i2\pi\eta x} dx = \int_{-1/2}^{1/2} e^{-i2\pi\eta x} dx \ .$$

Since $\phi_0(x)$ is symmetric, the transform $\Phi_0(\eta)$ is real and can easily be verified to equal

$$\Phi_0(\eta) = \frac{\sin\pi\eta}{\pi\eta} \quad . \tag{1.7}$$

This function, which is also referred to as "sinc" function in literature, is of particular importance in connection with Fourier transforms. It has zeroes at all integers $\eta = \pm 1, \pm 2, \ldots$ and assumes the value 1 at the origin $\eta = 0$ (Fig. 1.2) .

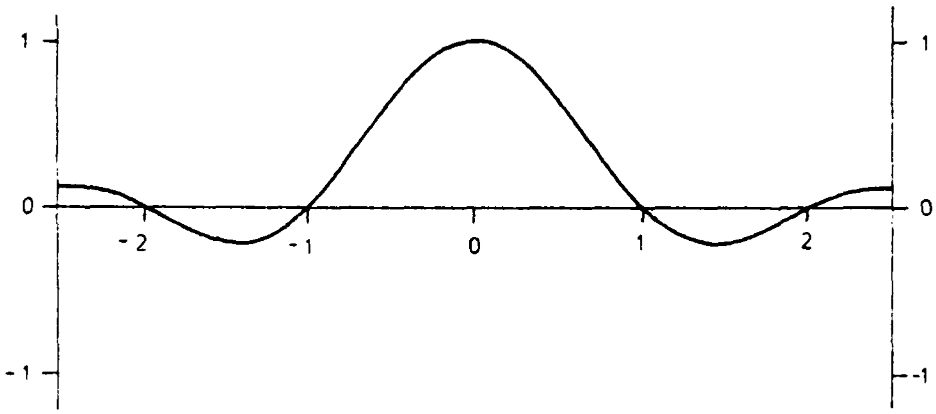

FIG. 1.2 The sinc - function

In full analogy to (1.5) and (1.6), the Fourier transform of an absolutely summable sequence $f = \{f_m\}$,

$$\sum_{-\infty}^{\infty} |f_m| < \infty \tag{1.5'}$$

is defined as

$$\bar{F}(\eta) = \sum_{-\infty}^{\infty} f_m e^{-i2\pi\eta m} \quad , \quad |\eta| < \frac{1}{2} \quad ; \tag{1.6a'}$$

its inverse assumes the form (Fuller, 1976, p. 115 ff.)

$$f_m = \int_{-1/2}^{1/2} \overline{F}(\eta)\, e^{i2\pi\eta m} d\eta \quad , \quad m = 0, \pm 1, \ldots \quad . \qquad (1.6b)'$$

In the following we show that the sequence $\{f_m\}$ represents a band-limited process.

According to equation (1.3b) the sequence $\{f_m\}$ is a sample of a continuous function $f(x)$, $f_m = f(m)$. This evaluation functional can also be expressed by

$$f_m = \int_{-\infty}^{\infty} f(x)\delta(x - m)dx$$

with $\delta(x)$ denoting the Dirac δ - distribution. The sequence $\{f_m\}$ can, consequently be expressed by

$$\{f_m\} = \int_{-\infty}^{\infty} f(x)\{\delta(x - m)\}dx \quad ,$$

which we simply write as

$$\overline{f}(x) = f(x) \cdot \sum_{-\infty}^{\infty} \delta(x - m) \quad , \qquad (1.8)$$

where the product is to be interpreted in the sense of distribution theory. The function

$$\Delta(x) = \sum_{-\infty}^{\infty} \delta(x - m) \qquad (1.9)$$

is called an "impulse comb", its Fourier transform can be shown to reproduce this impulse comb (Brigham, 1974, p. 19ff),

$$\Delta(\eta) = \sum_{-\infty}^{\infty} \delta(\eta - m) \quad . \tag{1.10}$$

According to the convolution theorem, a product of two functions in the space domain corresponds to a convolution of the corresponding transforms and vice versa. Therefore, the Fourier transform of $\bar{f}(x)$, defined in equation (1.8), is given by

$$\begin{aligned} \bar{F}(\eta) &= F(\eta) * \Delta(\eta) \quad ^{(1)} \\ &= \sum_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(\tau)\delta(\eta - m - \tau)d\tau \ , \end{aligned} \tag{1.11}$$

which finally equals

$$\bar{F}(\eta) = \sum_{-\infty}^{\infty} F(\eta - m) \quad . \tag{1.11'}$$

Consequently, $\bar{F}(\eta)$ is periodic with period equal to 1 and the corresponding frequencies range between $-\frac{1}{2} < \eta < \frac{1}{2}$ concluding the end of the proof that the sequence $\{f_m\}$ represents a band-limited process. Equation (1.6a)' can be verified by modifying (1.8) to

$$\bar{f}(x) = \sum_{-\infty}^{\infty} f_m \delta(x - m) \quad . \tag{1.8'}$$

Since a space shift by m corresponds to a multiplication by $e^{-i2\pi\eta m}$ in the frequency domain, and since the transform of the Dirac distribution equals 1 ,

$$\bar{F}(\eta) = \sum_{-\infty}^{\infty} f_m e^{-i2\pi\eta m}$$

(1) ... * denotes the convolution

follows immediately.

This result can be directly applied to the problem of calculating the Fourier transform of a step function as defined by equation (1.4) : the transform of the space shifted base function $\phi(x-m)$, with $\phi(x)$ defined by (1.1) and its transform by (1.7), equals

$$\frac{\sin\pi\eta}{\pi\eta} \cdot e^{-i2\pi\eta m},$$

and consequently, the Fourier transform of the step function (1.4) is given by

$$\hat{F}(\eta) = \frac{\sin\pi\eta}{\pi\eta} \sum_{-\infty}^{\infty} f_m e^{-i2\pi\eta m} . \tag{1.12}$$

Let us split up $\hat{F}(\eta)$ into a product

$$\hat{F}(\eta) = F_1(\eta) \cdot F_2(\eta) \tag{1.13a}$$

with

$$F_1(\eta) = \frac{\sin\pi\eta}{\pi\eta} \tag{1.13b}$$

and

$$F_2(\eta) = \sum_{-\infty}^{\infty} f_m e^{-i2\pi\eta m} . \tag{1.13c}$$

The following will be shown in the sequel:
Every local interpolation function has a spectrum of the form (1.13a) with $F_2(\eta)$ independent and $F_1(\eta)$ dependent on the interpolator.

$F_1(\eta)$ characterizes the interpolator base-function and will, therefore, be called the interpolator-characteristic; it acts as a frequency dampening function.

1.2 THE PIECEWISE LINEAR INTERPOLATION

Linear interpolation is a generally adopted tool for many different kinds of purposes. As far as the representation of two-dimensional surfaces is concerned, it is frequently employed in connection with a finite element representation. Here we are primarily interested in the linear interpolation of an infinite sequence of data located at the cardinal numbers and in its spectrum.

The base function is the continuous and strictly local function (Fig. 1.3)

$$\phi_1(x) = \begin{cases} 1 - |x| & \text{for} \quad |x| \leq 1 \\ 0 & \text{else} \end{cases} \quad . \tag{1.14}$$

It is an element out of K^1, the space of continuous functions having quadratically integrable first order derivatives. Its support has length 2, this is 1 unit more than that of the step base-function. The corresponding interpolant is simply given by the linear combination

$$\hat{f}(x) = \sum_{-\infty}^{\infty} f_m \phi_1(x - m) \quad . \tag{1.15}$$

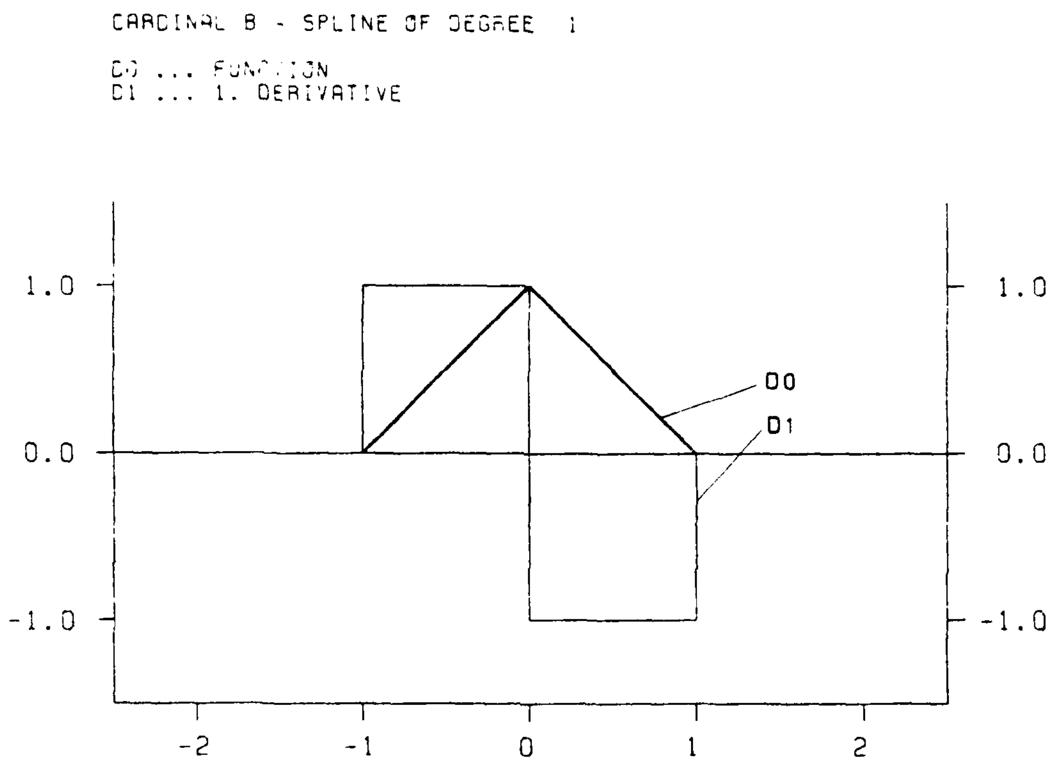

FIG. 1.3   Base function for K^1 .

Note that the coefficients of the linear combination are simply the function values at the grid points.

Can the base function for the linear interpolation be expressed by the step-base function? Yes, it can. The reader may verify the relation

$$\phi_1 = \phi_0 * \phi_0 \quad . \tag{1.16}$$

1.2.1 The spectrum of the piecewise linear interpolant

The Fourier transform of the base function (1.14) is given by

$$\Phi_1(\eta) = \int_{-1}^{1} (1 - |x|)e^{-i2\pi\eta x}dx$$

which can easily be shown to equal

$$\Phi_1(\eta) = \left(\frac{\sin\pi\eta}{\pi\eta}\right)^2 \quad . \tag{1.17}$$

Since ϕ_1 is the result of the convolution between ϕ_0 and ϕ_0, it follows from the convolution theorem that the corresponding Fourier transform is a simple product

$$\Phi_1 = \Phi_0 \cdot \Phi_0 \ ,$$

which is naturally fulfilled by equation (1.17) with Φ_0 given by (1.7).

Taking into account (1.13a,c) and (1.17), the Fourier transform of the piecewise linear and continuous interpolant is simply given by

$$\hat{F}(\eta) = \left(\frac{\sin\pi\eta}{\pi\eta}\right)^2 \sum_{-\infty}^{\infty} f_m e^{-i2\pi\eta m} \quad . \tag{1.18}$$

The piecewise linear and continuous interpolation function can be differentiated along the real line excluding the cardinal numbers:

$$D_x\hat{f}(x) = \sum_{-\infty}^{\infty} f_m D_x \phi_1(x - m)$$

with

$$D_x\phi_i(x) = \begin{cases} 1 & \text{for} \quad -1 < x < 0 \\ -1 & \text{for} \quad 0 < x < 1 \\ 0 & \text{else} \end{cases} \qquad (1.19)$$

In order to find the spectrum of $D_x\hat{f}$ we proceed as follows: the interpolant $\hat{f}$ equals the inverse Fourier transform

$$\hat{f}(x) = \int_{-\infty}^{\infty} \hat{F}(\eta)e^{i2\pi\eta x}d\eta \quad ;$$

we differentiate $\hat{f}$ and obtain

$$D_x\hat{f}(x) = i2\pi \int_{-\infty}^{\infty} \hat{F}(\eta)e^{i2\pi\eta x}\eta d\eta \quad ;$$

denoting the spectrum of $D\hat{f}$ by G, we find

$$G(\eta) = \int_{-\infty}^{\infty} D_x\hat{f}(x)\cdot e^{-i2\pi\eta x}dx$$

$$= i2\pi \int_{-\infty}^{\infty} [\int_{-\infty}^{\infty} \hat{F}(\eta)e^{i2\pi\eta x}\eta d\eta]e^{-i2\pi\mu x}dx \quad .$$

We integrate first with respect ot x ;

$$\int_{-\infty}^{\infty} e^{i2\pi x(\eta-\mu)}dx = \delta(\eta - \mu)$$

according to the theory of distributions (Brigham, 1974, p. 230); consequently,

$$G(\eta) = i2\pi \int_{\infty} \hat{F}(\eta)\eta\delta(\eta - \mu)d\eta \quad ,$$

and considering the properties of the Dirac distribution, we obtain for the spectrum of the differentiated piecewise linear and continuous function

$$G(\eta) = i2\pi\eta\hat{F}(\eta) \quad . \tag{1.20}$$

1.3 THE QUADRATIC SPLINE INTERPOLATION

Both the base function ϕ_0 for the step function "interpolation" and ϕ_1 for the piecewise linear interpolation are such that its support does not exceed two units. This is the reason why the coefficients of the linear combination are identical with the function values at the grid points. This is no longer true for a quadratic spline base function which has a support of 3 units. At the end of section 1.2 we have briefly stated that ϕ_1 is the result of a convolution between ϕ_0 and ϕ_0 . It is left as an exercise for the reader that

$$\phi_2 = \phi_1 * \phi_0 \tag{1.21}$$

with

$$\phi_2(x) = \frac{1}{2} \sum_{j=0}^{3} (-1)^j \binom{3}{j} (x + \frac{3}{2} - j)_+^2 \tag{1.22}$$

where

$$x_+^k = \begin{cases} x^k & \text{for} \quad x > 0 \\ 0 & \text{else} \end{cases} .$$

Explicitly written $\phi_2(x)$ is as follows

$$\phi_2(x) = \begin{cases} -x^2 + \frac{3}{4} & \text{for} \quad |x| \le \frac{1}{2} \\ \frac{1}{2}\left(\frac{3}{2} - |x|\right)^2 & \text{for} \quad \frac{1}{2} \le |x| \le \frac{3}{2} \\ 0 & \text{else} \end{cases} . \qquad (1.22)'$$

It is evident that the support of ϕ_2 is the open interval $(-\frac{3}{2}, \frac{3}{2})$. Of course, ϕ_2 is symmetric and assumes a maximum value of $\frac{3}{4}$ at the origin, $\phi_2(0) = \frac{3}{4}$. It can be easily verified that $\phi_2(x)$ is continuous and in addition, continuously differentiable. (Note that its first derivatives vanish at $x = -\frac{3}{2}$ and $x = \frac{3}{2}$.) It is, therefore, an element of K^2, the space of continuous and continuously differentiable functions having quadratically integrable second order derivatives. Its integral is one,

$$\int_{-3/2}^{3/2} \phi_2(x) = 1 .$$

Obviously ϕ_2 is not a sampling function ($\phi_2(k) \neq \delta_{k,0}$) which can be directly seen from Fig. 1.4. Therefore, it is not possible to express the interpolation function in a form like (1.4) or (1.15). With other words, the coefficients of the linear combination $\{g_m\}$

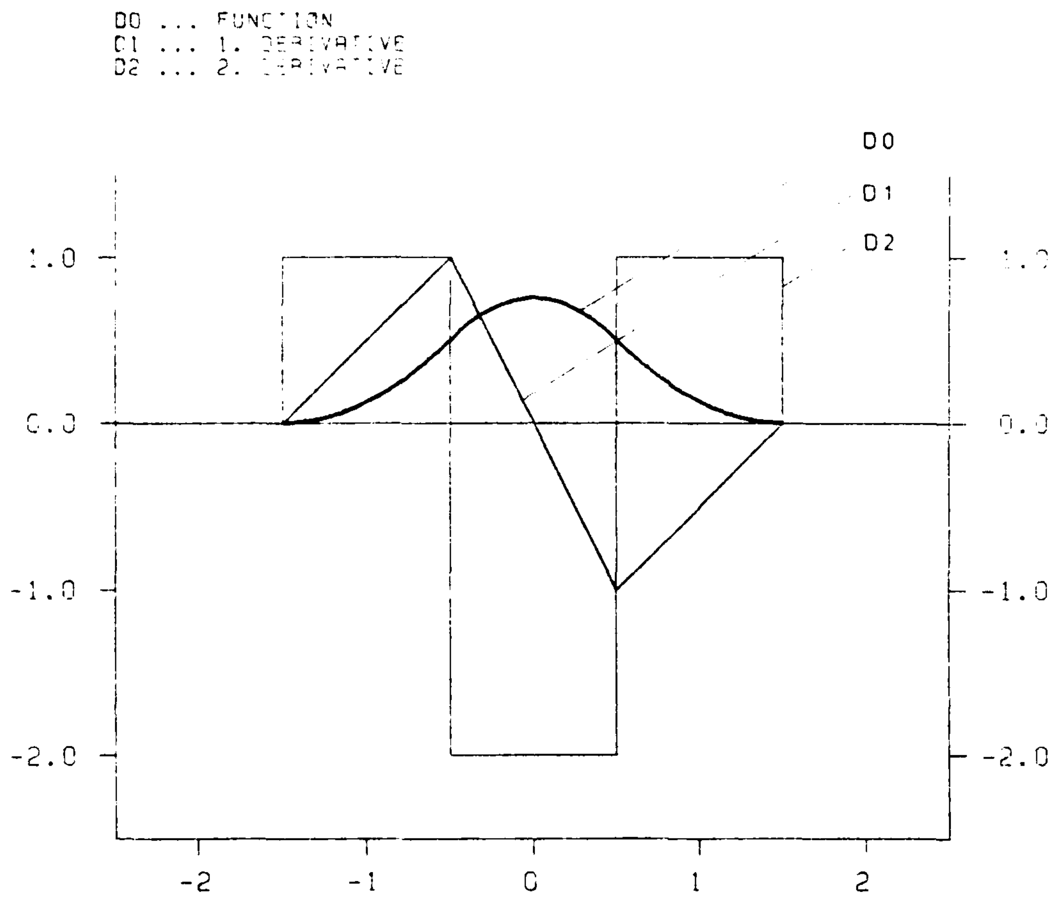

FIG. 1.4 Base function for K^2

$$\hat{f}(x) = \sum_{-\infty}^{\infty} g_m \phi_2(x - m) \tag{1.23}$$

do no longer equal the function values. They can, however, be expressed as a linear combination of the function values. This is shown as follows.

Let us introduce a <u>cardinal spline</u> or <u>sampling spline</u> $\psi_2(x)$ such that

$$\psi_2(k) = \delta_{k,0} \tag{1.24}$$

and

$$\hat{f}(x) = \sum_{-\infty}^{\infty} f_m \psi_2(x - m) \quad . \tag{1.25}$$

Then it should be possible to express ψ_2 in terms of ϕ_2 ,

$$\psi_2(x) = \sum_{-\infty}^{\infty} \alpha_k \phi_2(x - k) \quad . \tag{1.26}$$

The infinite vector of coefficients can be determined from the known properties of ϕ_2 and the properties of ψ_2 postulated by (1.24) : Evaluating (1.26) at all cardinal numbers $\{x = j\}$, $j = \ldots, -1, 0, 1, \ldots$ and observing (1.24), we obtain a linear system of infinite dimension,

$$A\alpha = e \quad . \tag{1.27}$$

Due to the compact support of ϕ_2 (see Fig. 1.4), the symmetric matrix A has only 3 non-vanishing diagonals (because ϕ_2 is different from zero at only 3 cardinal numbers). The main diagonal elements are constant and equal to $\phi_2(0) = \frac{3}{4}$, the elements of the other two diagonals are also constant and equal to $\phi_2(-1) = \phi_2(1) = \frac{1}{8}$. According to (1.24) ,

$$e = (\ \ldots, 0, 1, 0, \ldots\ ) \quad ; \tag{1.28}$$

$$\frac{1}{8}\begin{bmatrix} \ddots & \ddots & \ddots & \ddots & \ddots & & & \\ \ldots 0 & 1 & 6 & 1 & 0 \ldots & & & \\ & \ldots 0 & 1 & 6 & 1 & 0 \ldots & & \\ & & \ldots 0 & 1 & 6 & 1 & 0 \ldots & \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \cdot \begin{bmatrix} \vdots \\ \alpha_{-1} \\ \alpha_0 \\ \alpha_1 \\ \vdots \end{bmatrix} = \begin{bmatrix} \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix}$$

(1.27)'

Since the elements of A , a_{jk} , depend only on the difference $k-j$, it follows that A is a Toeplitz matrix of infinite dimension and, therefore, equivalent to a circulant matrix. Such kinds of systems are particularly easy to solve employing Fourier transform methods (Grenander and Szegö, 1958) : If $\bar{a}(\lambda)$ and $\bar{e}(\lambda)$ denote the discrete Fourier transforms of any row (or column) of A and the vector e , respectively, the solution vector α is simply given by the inverse Fourier transform of $\bar{e}(\lambda)$ / $\bar{a}(\lambda)$. (This is basically due to the fact that (1.27) represents a discrete convolution which reduces to a corresponding product in the frequency domain.)

$$\bar{a}(\lambda) = \frac{1}{8}(e^{-i\lambda} + 6 + e^{i\lambda}) = \frac{1}{4}(3 + \cos\lambda) \ ,$$

$$\bar{e}(\lambda) = 1 \ .$$

The inverse Fourier transform of $\bar{e}(\lambda)$ / $\bar{a}(\lambda)$ is simply

$$\alpha_k = \frac{4}{\pi} \int_0^{\pi} \frac{\cos k\lambda}{3+\cos\lambda} \, d\lambda \ ,$$

which can be shown to equal (Gradshteyn, 1971, p. 380, No. 3.613)

$$\alpha_k = \alpha_{-k} = \sqrt{2} \ (-3 + 2\sqrt{2})^{|k|} \ . \qquad (1.29)$$

With (1.29) the cardinal spline ψ_2 , shown in Fig. 1.5, can be expressed by

$$\psi_2(x) = \sqrt{2} \sum_{-\infty}^{\infty} (-3 + 2\sqrt{2})^{|k|} \phi_2(x - k) \ . \qquad (1.26)'$$

This infinite sum can be reduced to a finite sum consisting of

only 3 terms if we take into account the properties of the base function ϕ_2 : ϕ_2 has a compact support which is the open interval $(-\frac{3}{2}, \frac{3}{2})$ (see Fig. 1.4), ϕ_2 is symmetric $(\phi_2(-x) = = \phi_2(x))$. Observing the symmetry of α_k, $\alpha_{-k} = \alpha_k$, equation (1.26)' reduces to

$$\psi_2(x) = \sum_{k=k_1}^{k_1+2} \alpha_k \phi_2(|x| - k)$$

with (1.26)"

$$k_1 = \mathrm{int}(|x| + 0.5) - 1 \quad .$$

The verification that $\psi_2(x)$ is in fact a sampling function fulfilling equation (1.24), is left as an exercise for the reader.

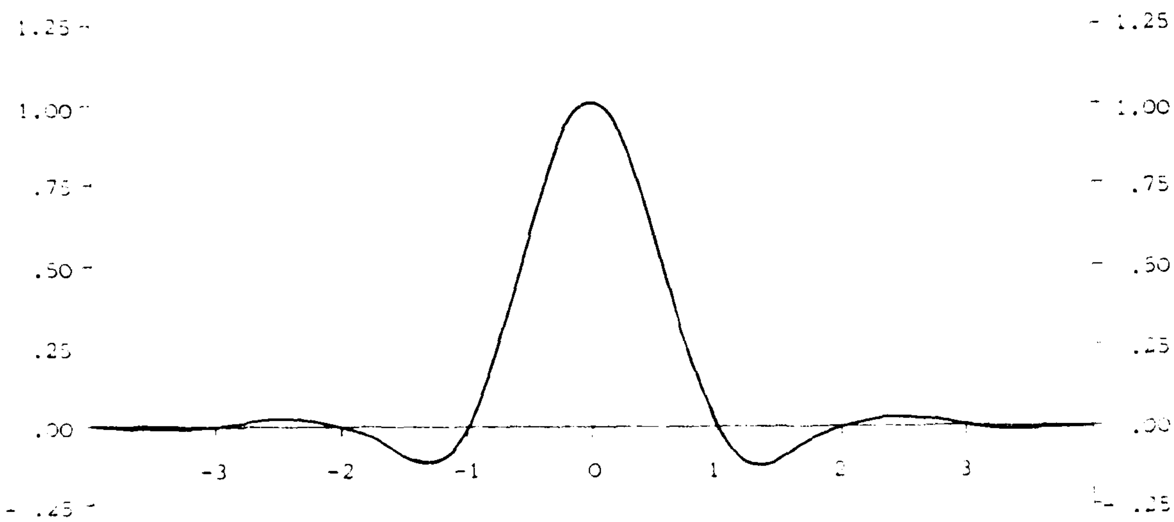

FIG. 1.5 Cardinal spline $\psi_2(x)$

It seems to be worthwile to point out that ψ_2 has no longer a finite support. This is somewhat astonishing since the cardinal sampling function of the space K^1, denoted by ψ_1, had still a finite support. This is basically due to the fact that the inverse of the tri-diagonal transformation matrix A is a full infinite matrix for K^2 ; the corresponding matrices for K^0 and K^1 are identity matrices : the base functions ϕ_0 and ϕ_1 fulfil equation (1.24) and, therefore, these base functions are already sampling functions. Furthermore, the sum of all α_k's equals 1

$$\sum_{-\infty}^{\infty} \alpha_k = \sqrt{2}\,\{1 + 2 \sum_{1}^{\infty} (-3 + 2\sqrt{2})^k\} = 1 \ , \qquad (1.30)$$

and, therefore, considering the fact that the integral of ϕ_2 is unity, also the integral of ψ_2 equals 1 ,

$$\int_{-\infty}^{\infty} \psi_2(x)\,dx = 1 \ .$$

1.3.1 The spectrum of the quadratic spline interpolant

The Fourier transform of the quadratic spline interpolant (equation (1.25)) ,

$$\hat{f}(x) = \sum_{-\infty}^{\infty} f_m \psi_2(x - m)$$

is given by

$$\hat{F}(\eta) = \sum_{-\infty}^{\infty} f_m \int_{-\infty}^{\infty} \psi_2(x - m) e^{-i2\pi\eta x} dx \ ,$$

which, according to the translation property of Fourier transforms reduces to

$$\hat{F}(\eta) = \int_{-\infty}^{\infty} \psi_2(x) e^{-i2\pi\eta x} dx \cdot \sum_{-\infty}^{\infty} f_m e^{-i2\pi\eta m} \quad .$$

Considering the definition of the cardinal sampling function ψ_2 given by equation (1.26) and taking into account the symmetry of ψ_2 , the above integral can be written as

$$\int_{-\infty}^{\infty} \psi_2(x) e^{-i2\pi\eta x} dx = \Phi_2(\eta) \sum_{-\infty}^{\infty} \alpha_k \cos(2\pi\eta k) \quad . \tag{1.31}$$

Here $\Phi_2(\eta)$ denotes the Fourier transform of the base function $\phi_2(x)$. Since ϕ_2 is the result of a convolution between ϕ_1 and ϕ_0 (equation 1.21), it follows from the convolution theorem that Φ_2 is a product of Φ_1 and Φ_0 ; Φ_1 in turn is a product of Φ_0 and Φ_0 with Φ_0 given by equation (1.7). Therefore,

$$\Phi_2(\eta) = \Phi_0^3(\eta) = \left(\frac{\sin\pi\eta}{\pi\eta}\right)^3 \quad . \tag{1.32}$$

The infinite sum in equation (1.31) needs still to be calculated. Since α_k and the cosine function are symmetric and moreover, according to equation (1.29)

$$\alpha_k = \sqrt{2}\,\alpha^k$$

with

$$\alpha = -3 + 2\sqrt{2} \quad ,$$

we can write

$$\sum_{-\infty}^{\infty} \alpha_k \cos 2\pi\eta k = \sqrt{2}\,(1 + 2 \sum_{1}^{\infty} \alpha^k \cos 2\pi\eta k) \quad ,$$

and due to (Gradshteyn, 1971, p.54, No. 1.447) the infinite sum can be expressed in a closed form such as

$$\sum_{-\infty}^{\infty} \alpha_k \cos 2\pi\eta k = \sqrt{2}\, \frac{1 - \alpha^2}{1-2\alpha\cos 2\pi\eta+\alpha^2} \quad . \tag{1.33}$$

With α as defined above this equation can, after trivial algebraic operations, be reduced to

$$\sum_{-\infty}^{\infty} \alpha_k \cos 2\pi\eta k = \frac{4}{3+\cos 2\pi\eta} \quad . \tag{1.33'}$$

With (1.31) we finally obtain the Fourier transform of the quadratic spline interpolant

$$\hat{F}(\eta) = \frac{4}{3+\cos 2\pi\eta} \left(\frac{\sin\pi\eta}{\pi\eta}\right)^3 \sum_{-\infty}^{\infty} f_m e^{-i2\pi\eta m} \quad , \tag{1.34}$$

which is again of the form (1.13a).

The quadratic spline is continuous and continuously differentiable with quadratically integrable second order derivatives. The spectrum of its first and second order derivative can be derived in complete analogy to subsection 1.2.1. We obtain for the spectrum of $D_x\hat{f}(x)$

$$G_1(\eta) = i2\pi\eta\hat{F}(\eta) \tag{1.35a}$$

and for the spectrum of $D_x^2\hat{f}(x)$

$$G_2(\eta) = (i2\pi\eta)^2 \hat{F}(\eta) \tag{1.35b}$$

with $\hat{F}(\eta)$ given by (1.34).

1.4 THE CUBIC SPLINE INTERPOLATION

Let us convolve the base function ϕ_2 , defined by equation (1.22)' and shown in Fig. 1.4, with the base function ϕ_0 , defined by equation (1.1) and shown in Fig. 1.1,

$$\phi_3 = \phi_2 * \phi_0 \quad . \tag{1.36}$$

It can be shown (Schoenberg, 1973, p. 11) that ϕ_3 satisfies the following equation

$$\phi_3(x) = \frac{1}{6} \sum_{j=0}^{4} (-1)^j \binom{4}{j} (x + 2 - j)_+^3 \tag{1.37}$$

where

$$x_+^k = \begin{cases} x^k & \text{for} \quad x > 0 \\ 0 & \text{else} \end{cases} \quad .$$

Explicitly written $\phi_3(x)$ is as follows:

$$\phi_3(x) = \frac{1}{6} \begin{cases} 3|x|^3 - 6x^2 + 4 & \text{for} \quad |x| \le 1 \\ (2 - |x|)^3 & \text{for} \quad 1 \le |x| \le 2 \\ 0 & \text{else} \end{cases} \quad . \tag{1.37'}$$

It is obvious that the support of Φ_3 is the open interval $(-2,2)$, its length is, therefore, equal to 4 . Of course, Φ_3 is symmetric and assumes a maximum value of $\frac{2}{3}$ at the origin, $\Phi_3(0) = \frac{2}{3}$. It can be easily verified that Φ_3 is continuous and twice continuously differentiable. (Note that Φ_3 , its first and second order derivatives vanish at $x = -2$ and $x = 2$.) Therefore, Φ_3 is an element of K^3 , the space of continuous and twice continuously differentiable functions having quadratically integrable third order derivatives. Its integral is one, Φ_3 is a base function for the cubic spline interpolation, but unfortunately not a sampling function ($\Phi_3(k) \neq \delta_{k,0}$) which can be seen from Fig. 1.6. Therefore, it is not possible to express the interpolation function in a form like (1.4) or (1.15); we are in a similar situation as we have been with Φ_2 : the coefficents of the linear combination are no longer the function values themselves; however, they can be expressed as linear combinations of the function values at the cardinal numbers.

The derivation of the cardinal spline or sampling spline ψ_3 is practically identical to the derivation of ψ_2 as presented in section 1.3. Therefore, we point out only the important steps.

The <u>cardinal cubic spline</u> $\psi_3(x)$, fulfilling equation (1.39), is defined by

$$\psi_3(x) = \sum_{-\infty}^{\infty} a_k \Phi_3(x - k) \quad . \tag{1.38}$$

The infinite vector of coefficients $\{a_k\}$ can be determined from the known properties of the base function Φ_3 and the sampling properties of ψ_3 postulated by

$$\psi_3(k) = \delta_{k,0} \quad . \tag{1.39}$$

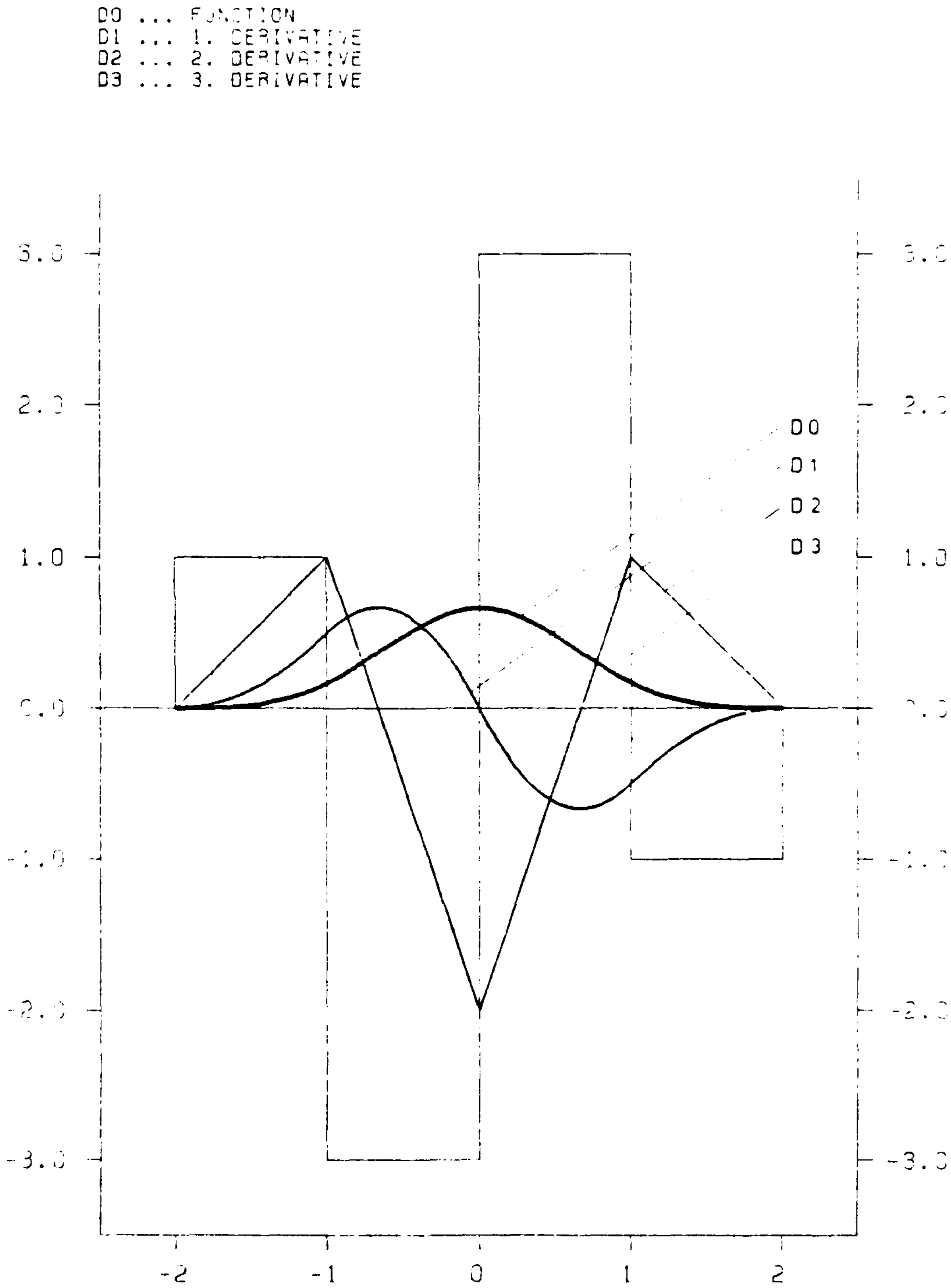

FIG. 1.6 Base function for K^3

Evaluating (1.38) at all cardinal numbers and observing (1.39), we obtain a linear system of infinite dimension,

$$A\alpha = e \ ,$$

with α and e denoting the infinite vector of coefficients $\{\alpha_k\}$ and e the infinite vector given by equation (1.28). ϕ_3 has finite support of length equal to 4 (see Fig. 1.6) and does not vanish at the cardinal numbers $x = -1, 0, 1$. Therefore, the symmetric matrix A has only 3 non-vanishing diagonals. The main diagonal elements are constant and equal to $\phi_3(0) = \frac{2}{3}$, the elements of the other two diagonals are also constant and equal to $\phi_3(-1) = \phi_3(1) = \frac{1}{6}$,

$$\frac{1}{6}\begin{bmatrix} \ddots & \ddots & \ddots & \ddots & \ddots & & & \\ \cdots 0 & 1 & 4 & 1 & 0 \cdots & & & \\ & \cdots 0 & 1 & 4 & 1 & 0 \cdots & & \\ & & \cdots 0 & 1 & 4 & 1 & 0 \cdots & \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \cdot \begin{bmatrix} \vdots \\ \alpha_{-1} \\ \alpha_0 \\ \alpha_1 \\ \vdots \end{bmatrix} = \begin{bmatrix} \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix} . \qquad (1.40)$$

The Elements of A, a_{jk}, depend only on the difference $k - j$; therefore, A is a Toeplitz matrix of infinite dimension and equivalent to a circulant matrix. The solution vector is found by Fourier transform methods in the very same way as it was obtained in section 1.3 : the Fourier transform of a row (or column) of A is given by

$$\bar{a}(\lambda) = \frac{1}{6}\left(e^{-i\lambda} + 4 + e^{i\lambda}\right) = \frac{1}{3}(2 + \cos\lambda) \ .$$

The solution vector α is obtained by an inverse Fourier trans-

form of $\bar{e}(\lambda)/\bar{a}(\lambda)$ with $\bar{e}(\lambda) = 1$ as in the foregoing section,

$$\alpha_k = \frac{3}{\pi} \int_0^{\pi} \frac{\cos k\lambda}{2+\cos\lambda} \, d\lambda \quad ,$$

which can be shown (Gradshteyn, 1971, p. 380, No. 3.613) to equal

$$\alpha_k = \alpha_{-k} = \sqrt{3}\ (-2 + \sqrt{3})^{|k|} \quad . \qquad (1.41)$$

With (1.41) the cardinal cubic spline ψ_3 , shown in Fig. 1.7, can be expressed by

$$\psi_3(x) = \sqrt{3} \sum_{-\infty}^{\infty} (-2 + \sqrt{3})^{|k|} \phi_3(x - k) \quad . \qquad (1.38)'$$

The infinite sum reduces to a finite one of only 4 elements, when taking into account the properties of the base function ϕ_3 :

$$\psi_3(x) = \sum_{k=k_1}^{k_1+3} \alpha_k \phi_3(|x| - k) \qquad (1.38)''$$

with

$$k_1 = \mathrm{int}(|x|) - 1 \quad .$$

The reader is invited to verify that $\psi_3(x)$ is in fact a sampling function fulfilling equation (1.39). The reasoning for the unlimited support of ψ_3 can be carried over from section 1.3 without any change.

This sum of all coefficients α_k can be verified to equal 1,

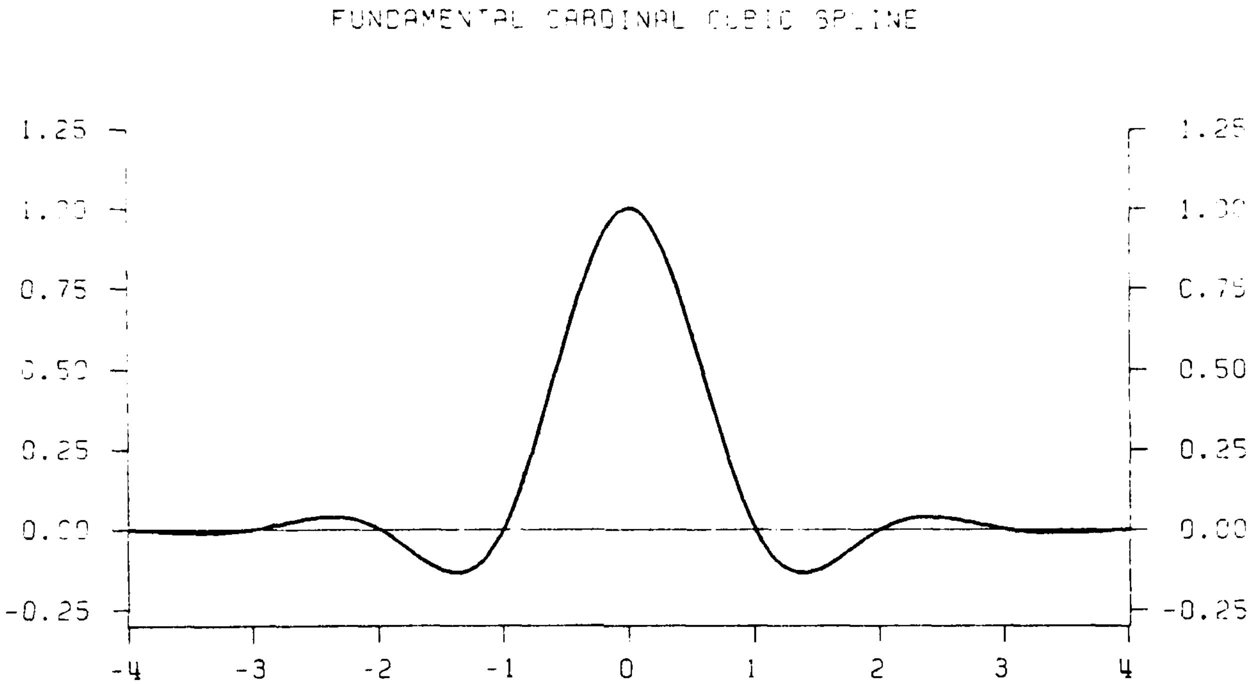

FIG. 1.7 Cardinal spline $\psi_3(x)$

$$\sum_{-\infty}^{\infty} \alpha_k = \sqrt{3}\,[1 + 2 \sum_{1}^{\infty} (-2 + \sqrt{3})^k] = 1 \quad ; \qquad (1.42)$$

since the integral of ϕ_3 is unity, it follows from the definition of ϕ_3 (cf. equation (1.38)) that the integral of ψ_3 is also 1 ,

$$\int_{-\infty}^{\infty} \psi_3(x)\,dx = 1 \quad .$$

1.4.1 The spectrum of the cubic spline interpolant

The Fourier transform of the cubic spline interpolant

$$\hat{f}(x) = \sum_{-\infty}^{\infty} f_m \psi_3(x - m)$$

is given by

$$\hat{F}(\eta) = \sum_{-\infty}^{\infty} f_m \int_{-\infty}^{\infty} \psi_3(x - m) e^{-i2\pi\eta x} dx \quad ,$$

which, according to the translation property of Fourier transforms, reduces to

$$\hat{F}(\eta) = \int_{-\infty}^{\infty} \psi_3(x) e^{-i2\pi\eta x} dx \cdot \sum_{-\infty}^{\infty} f_m e^{-i2\pi\eta x} .$$

Considering the definition of the cardinal sampling function ψ_3 given by equation (1.38) and taking into account the symmetry of ψ_3 , the above integral can be written as

$$\int_{-\infty}^{\infty} \psi_3(x) e^{-i2\pi\eta x} dx = \Phi_3(\eta) \sum_{-\infty}^{\infty} \alpha_k \cos(2\pi\eta k) \quad . \qquad (1.43)$$

Here $\Phi_3(\eta)$ denotes the Fourier transform of the base function $\phi_3(x)$. Since $\phi_3(x)$ is the result of a convolution between ϕ_2 and ϕ_0 (equation 1.36) , it follows from the convolution theorem that Φ_3 is a product of Φ_2 and Φ_0 ; with (1.7) and (1.32) Φ_3 is simply given by

$$\Phi_3(\eta) = \Phi_0^4(\eta) = \left(\frac{\sin\pi\eta}{\pi\eta}\right)^4 . \qquad (1.44)$$

Since both α_k and the cosine function are symmetric and because of

$$\alpha_k = \sqrt{3}\ \alpha^k$$

with

$$\alpha = -2 + \sqrt{3} \quad ,$$

we can write

$$\sum_{-\infty}^{\infty} \alpha_k \cos 2\pi\eta k = \sqrt{3}\ (1 + 2 \sum_{1}^{\infty} \alpha^k \cos 2\pi\eta k) \quad ,$$

and due to (Gradshteyn, 1971, p.54, No. 1.447) the infinite sum can be replaced by a closed expression such as

$$\sum_{-\infty}^{\infty} \alpha_k \cos 2\pi\eta k = \sqrt{3}\ \frac{1 - \alpha^2}{1-2\alpha\cos 2\pi\eta+\alpha^2} \quad .$$

With α as above this equation reduces to

$$\sum_{-\infty}^{\infty} \alpha_k \cos 2\pi\eta k = \frac{3}{2+\cos 2\pi\eta} \quad . \qquad (1.45)$$

With (1.43) we finally obtain the Fourier transform of the cubic spline interpolant

$$\hat{F}(\eta) = \frac{3}{2+\cos 2\pi\eta}\left(\frac{\sin\pi\eta}{\pi\eta}\right)^4 \sum_{-\infty}^{\infty} f_m e^{-i2\pi\eta m} \quad , \tag{1.46}$$

which is again of the form (1.13a).

The cubic spline is continuous and twice continuously differentiable with quadratically integrable third order derivatives. The spectra of these spline derivatives can be obtained in complete analogy to subsection 1.2.1. We obtain for the spectrum of $D_x^k \hat{f}(x)$

$$G_k(\eta) = (i2\pi\eta)^k \hat{F}(\eta) \quad , \qquad k = 0, \ldots, 3 \; . \tag{1.47}$$

1.5 HIGHER AND HIGHEST ORDER INTERPOLANTS

In the foregoing sections we have been dealing with low order interpolants on the sequence of cardinal numbers. All these functions turned out to be closely related to each other. Before we present a condensed form of the whole array of relations, we would like to draw attention to interpolation functions of higher order continuous differentiability; we shall end up with the highest order interpolant possible.

It was shown in the last sections that the base function for the space K^0 is the step function ϕ_0 as defined by (1.1), and that the base functions for higher order spaces result from convolutions of ϕ_0's . In general, the base function ϕ_n for the space K^n is given by

$$\phi_n = \underbrace{\phi_0 * \phi_0 * \ldots * \phi_0}_{n\text{ - times}} . \tag{1.48}$$

According to the convolution theorem, the corresponding Fourier transform Φ_n is an n-fold product of Φ_0 , the Fourier transform of ϕ_0 ,

$$\Phi_n = \underbrace{\Phi_0 \cdot \Phi_0 \cdot \ldots \cdot \Phi_0}_{n\text{ - times}} = \Phi_0^{n+1} . \tag{1.49}$$

The base function ϕ_0 has a support of 1 unit ; each convolution, therefore contributes to an increase of the support by 1 unit; consequently, the support of ϕ_n equals $n + 1$ units,

$$\text{support}(\phi_n) = n + 1 . \tag{1.50}$$

The Fourier transform Φ_0 is the function

$$\Phi_0(n) = \frac{\sin\pi n}{\pi n} ;$$

therefore, the Fourier transform Φ_n is simply given by

$$\Phi_n(n) = \left(\frac{\sin\pi n}{\pi n}\right)^{n+1} . \tag{1.51}$$

The interpolating function, as an element of K^n , can be expressed as a linear combination of the form

$$\hat{f}_n(x) = \sum_{-\infty}^{\infty} f_m \psi_n(x - m) \tag{1.52}$$

with ψ_n being a cardinal spline of degree n and $\{f_m\}$ the infinite data vector. The function ψ_n , in turn, is a linear

combination of base functions ϕ_n ,

$$\psi_n(x) = \sum_{-\infty}^{\infty} \alpha_k \phi_n(x - k) \quad . \tag{1.53}$$

The infinite vector $\{\alpha_k\}$ is determined from the sampling property of ψ_n ,

$$\psi_n(k) = \delta_{k,0} \, , \tag{1.54}$$

and the properties of ϕ_n ; it leads to the solution of the infinite linear system

$$A\alpha = e \tag{1.55}$$

with

$$\alpha^T = [\ldots \alpha_{-1}, \alpha_0, \alpha_1, \ldots]$$

and

$$e^T = [\ldots 0 \; , \; 1 \; , \; 0, \ldots] \; .$$

The matrix A has very special properties: it is an infinite symmetric matrix of band structure; all diagonals have identical elements. Such a matrix is known as a Toeplitz matrix which is equivalent to a circulant matrix due to its infinite dimension.

The bandwidth N (= number of non-vanishing diagonals) depends only on the support of the base function ϕ_n :

$$N = \begin{cases} n+1 & \text{if} \quad n \quad \text{is even} \\ n & \text{if} \quad n \quad \text{is odd} \end{cases} \quad . \tag{1.56}$$

E.g. : For the linear interpolation $(n = 1)$, A is a pure diagonal matrix, moreover, it is the unit matrix, therefore $\alpha = e$ and, as a consequence (equation (1.53)) , the original base

function ϕ_1 and the corresponding sampling function ψ_1 coincide.

Denoting a row (or column) of A by a, $a = \{a_k\}$, the elements $\{a_k\}$ are given by

$$a_k = \phi_n(k) \quad . \tag{1.57}$$

The inverse of A can be found via Fourier transform methods : Denoting the discrete Fourier transform of a by $\bar{a}$,

$$\begin{aligned}\bar{a}(\lambda) &= \sum_{-\infty}^{\infty} a_k e^{-ik\lambda} \\ &= \sum_{-\infty}^{\infty} \phi_n(k)\cos k\lambda \\ &= \phi_n(0) + 2 \sum_{1}^{\mathrm{int}(n/2)} \phi_n(k)\cos k\lambda \ ,\end{aligned} \tag{1.58}$$

and by

$$\bar{e}(\lambda) = \sum_{-\infty}^{\infty} e_k e^{-ik\lambda} = 1 \quad ,$$

the coefficient vector of unknowns $\{\alpha_k\}$ can be directly found by the inverse Fourier tranform

$$\begin{aligned}\alpha_k &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\bar{e}(\lambda)}{\bar{a}(\lambda)} e^{ik\lambda} d\lambda \\ &= \frac{1}{\pi} \int_{0}^{\pi} \frac{1}{\bar{a}(\lambda)} \cos k\lambda \, d\lambda \quad .\end{aligned} \tag{1.59}$$

This integral plays a central role within the framework of splines :

if only 1 frequency is represented in $\bar{a}(\cdot)$ (apart from the zero frequency) , equation (1.59) admits a closed expression and the α_k's can be found directly $(n \leq 3)$; if more than 1 frequency is represented $(n > 3)$, this simplicity is no longer present.

The cardinal spline of degree n , ψ_n , as defined by equation (1.53), has unlimited support for $n > 1$; the higher n , the weaker is its tendency to tend to zero with increasing argument, and therefore, the less is its local behavior. Its spectrum has the general form

$$\hat{F}_n(\eta) = H_n(\eta) \left(\frac{\sin\pi\eta}{\pi\eta}\right)^{n+1} \sum_{-\infty}^{\infty} f_m e^{-i2\pi\eta m} \qquad (1.60)$$

where $H_n(\eta)$ is characteristic for the order n . For $n = 0, 1$ $H_n = 1$, for $n = 2, 3$ H_n is given by (1.33)' and (1.45). Explicitly

$$H_n(\eta) = \sum_{-\infty}^{\infty} \alpha_k \cos 2\pi\eta k \qquad (1.61)$$

with $\{\alpha_k\}$ denoting the vector of coefficients for the degree n in consideration. These functions $H_n(\eta)$ show the very interesting property of compensating the dampening function $(\sin\pi\eta/\pi\eta)^{n+1}$ better and better with increasing n .

Let us put now the obvious question:

What kind of function is the limit cardinal spline

$$\lim_{n\to\infty} \psi_n(x) \quad ?$$

Obviously

$$\hat{f}_\infty(x) = \sum_{-\infty}^{\infty} f_m \psi_\infty(x - m)$$

should interpolate the data vector $\{f_m\}$ which is supposedly an element of l_2^∞ , the space of quadratically summable finite differences of all orders k ,

$$l_2^\infty := \bigcup_{k=0}^{\infty} l_2^k .$$

The cardinal function itself should correspondingly be an element of K_2^∞ , the space of quadratically integrable derivatives of all orders k ,

$$K_2^\infty := \bigcup_{k=0}^{\infty} K_2^k .$$

There is a one-to-one correspondence between the spaces l_2^∞ and K_2^∞ (Schoenberg, 1973). The unit-sequence (..., 0, 1, 0, ...) is surely an element of l_2^∞ . It can be shown (Schoenberg, 1973) that the function

$$\frac{\sin \pi x}{\pi x}$$

is the only element of K_2^∞ which interpolates the unit sequence. Therefore, we conclude that

$$\lim_{n \to \infty} \psi_n(x) = \frac{\sin \pi x}{\pi x} \quad . \tag{1.62}$$

The corresponding limit interpolation function is

$$\hat{f}_\infty(x) = \sum_{-\infty}^{\infty} f_m \frac{\sin\pi(x-m)}{\pi(x-m)} \quad . \tag{1.63}$$

ψ_∞ can be said to be the cardinal spline of highest possible degree (∞) , $\hat{f}_\infty$ the corresponding interpolant. Its tendency to approach zero is rather weak and, therefore, its interpolation behavior cannot be considered local. (Fig. 1.8).

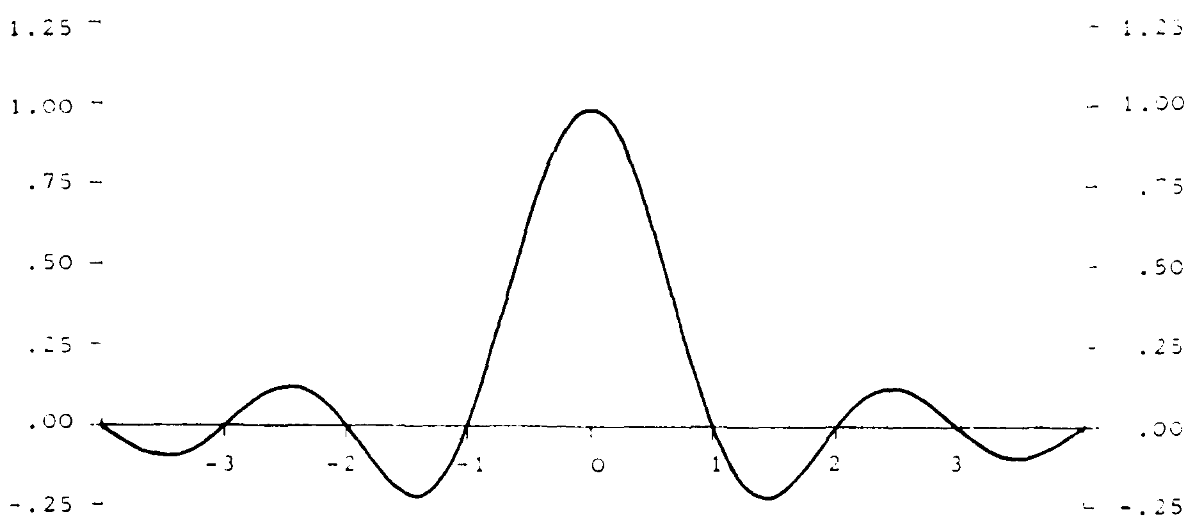

FIG. 1.8 Cardinal spline $\psi_\infty(x)$

The spectrum of ψ_∞ can easily be shown to be a unit step function. Actually it has already been shown in subsection 1.1.1 that the spectrum of the unit step function equals $\sin\pi\omega/\pi\omega$,

$$\Psi_\infty(\eta) = \int_{-\infty}^{\infty} \frac{\sin \pi x}{\pi x}\, e^{-i2\pi\eta x} dx = \begin{cases} 1 & \text{for} \quad |\eta| < \frac{1}{2} \\ \frac{1}{2} & \text{for} \quad |\eta| = \frac{1}{2} \\ 0 & \text{else} \end{cases} \quad . \tag{1.64}$$

(Gradshteyn, 1971, p. 428, No. 3.741). Since $\Psi_\infty(\eta) = 0$ for all $|\eta| > \frac{1}{2}$, we conclude that ψ_∞ is a low pass filter - the interpolation function $\hat{f}_\infty$ does not contain frequencies higher than $|\eta| > \frac{1}{2}$. With (1.64) and the translation property taken into account, the Fourier transform of $\hat{f}_\infty$ is given by

$$\hat{F}_\infty(\eta) = \begin{cases} \sum_{-\infty}^{\infty} f_m e^{-i2\pi\eta m} & \text{for} \quad |\eta| < \frac{1}{2} \\ 0 & \text{for} \quad |\eta| > \frac{1}{2} \end{cases} \tag{1.65}$$

(Here we did not consider the case $|\eta| = \frac{1}{2}$.) The relation between ψ_∞ and Ψ_0 is quite remarkable and deserves special attention:

The cardinal spline of highest possible degree equals the spectrum of the cardinal spline of lowest possible degree and vice versa.

$$\begin{aligned} \psi_\infty &= \Psi_0 \\ \psi_0 &= \Psi_\infty \quad . \end{aligned} \tag{1.66}$$

1.5.1 Guidelines for cardinal spline interpolation of arbitrary degree

The detailed and straightforward presentation in sections 1.1 to 1.4 will be complemented by some additional and quite useful relations; the procedures involved starting from the choice of an appropriate base function and ending with the interpolation formula are presented in a flowchart-like diagram.

a) The first additional information is related to the function values of a base function ϕ_n at the cardinal numbers. Remember that they represent the entries of the transformation matrix A (equations (1.55) and (1.57)) which, in turn, provides the link between the base function and the corresponding sampling function, therefore, they simply need to be known.

Basically there are two approaches, a direct one which derives the function values employing Peano's theorem (Schoenberg, 1973, p. 2 ff.) and an indirect one, which, according to the author, is simpler and more transparent:

We know from (1.51) that the Fourier transform of the base function ϕ_n is the $(n+1)$th power of $\sin\pi\eta/\pi\eta$,

$$\Phi_n(\eta) = \left(\frac{\sin\pi\eta}{\pi\eta}\right)^{n+1} .$$

The base function ϕ_n , therefore, is obtained by the inverse Fourier transform of Φ_n ,

$$\phi_n(x) = 2\int_0^\infty \Phi_n(\eta)\cos 2\pi\eta x\,d\eta .$$

Consequently, the values of the base function at the cardinal numbers is simply obtained by replacing the variable x by an integer k ,

$$\phi_n(k) = 2\int_0^{\infty}\left(\frac{\sin\pi\eta}{\pi\eta}\right)^{n+1}\cos 2\pi k\eta\, d\eta \quad . \tag{1.67}$$

With (Gradshteyn, 1971, p. 471, No. 3.836) this integral can be expressed as

$$\phi_n(k) = \begin{cases} \dfrac{1}{n!}\displaystyle\sum_{j=0}^{\mathrm{int}\left(\frac{n}{2}\right)+|k|}(-1)^j\binom{n+1}{j}\left(\frac{n+1}{2}+|k|-j\right)^n & \text{for } 0 \le |k| \le \mathrm{int}\left(\frac{n}{2}\right) \\ 0 & \text{for } |k| > \mathrm{int}\left(\frac{n}{2}\right) \end{cases} \tag{1.68}$$

It can easily be shown to be equivalent to an expression obtained by Schoenberg (1973)

$$\phi_n(k) = \frac{1}{n!}\sum_{j=0}^{n+1}(-1)^j\binom{n+1}{j}\left(\frac{n+1}{2}+k-j\right)_+^n \quad , \tag{1.69}$$

taking into account the definition of

$$x_+^n = \begin{cases} x^n & \text{for } x > 0 \\ 0 & \text{else} \end{cases} \quad .$$

Either (1.68) or (1.69) allows us to verify the identity

$$\sum_{k=-\infty}^{\infty} \phi_n(k) = 1 \qquad \forall\ n \in \mathbb{N}_0 \quad . \tag{1.70}$$

b) The second additional information concerns the calculation of the cardinal spline sampling functions from the corresponding base functions. In all foregoing sections we have presented one way of obtaining the sampling functions which can be sketched as follows:

$$\{a_k\} = \{\phi_n(k)\}$$
$$\downarrow$$
$$\bar{a}(\eta) = \sum_{-\infty}^{\infty} a_k \cos 2\pi k\eta$$
$$\downarrow$$
$$\alpha_k = 2\int_0^{\pi} \frac{1}{\bar{a}(\eta)} \cos 2\pi k\eta d\eta$$
$$\downarrow$$
$$\psi_n(x) = \sum_{-\infty}^{\infty} \alpha_k \phi_n(x-k) \quad .$$

The other way of obtaining ψ_n avoids the explicit calculation of the infinite vector $\alpha = \{\alpha_k\}$; this and the above (at least theoretically) infinite summation can be replaced by an inverse Fourier transform. Observe that the Fourier transform of ψ_n is given by

$$\Psi_n(\eta) = \Phi_n(\eta) \sum_{-\infty}^{\infty} \alpha_k \cos 2\pi k\eta \ ;$$

according to (1.55)

$$\alpha = A^{-1} e \quad ,$$

which transforms to

$$\sum_{-\infty}^{\infty} \alpha_k \cos 2\pi k\eta = \left\{\sum_{-\infty}^{\infty} a_k \cos 2\pi k\eta\right\}^{-1} = \bar{a}^{-1}(\eta) \quad ;$$

(note that the vector α represents a row or column of A^{-1}). Taking this relation into account, the Fourier transform of ψ_n is given by

$$\Psi_n(\eta) = \frac{\Phi_n(\eta)}{\bar{a}(\eta)} \tag{1.71}$$

and the cardinal spline of degree n by its inverse Fourier transform

$$\psi_n(x) = 2\int_0^{\infty} \frac{\left(\frac{\sin\pi\eta}{\pi\eta}\right)^{n+1} \cos 2\pi\eta x d\eta}{\sum_{-\infty}^{\infty} \phi_n(k) \cos 2\pi k\eta} \quad . \tag{1.71'}$$

We have thus avoided the actual calculation of the coefficients $\{\alpha_k\}$. This representation shows very clearly that only the basis spline's function values at the cardinal numbers are required to obtain the corresponding cardinal spline; it has, however, no practical consequences, since it is much easier to perform a Fourier transform (for the cardinal numbers only) once for all, and then to combine the resulting α_k's linearly with the base functions as in (1.53). We will need (1.71) for a different reason in the next chapter.

The following graphs illustrate quite clearly the relations between spline base functions, their Fourier transforms, and corresponding cardinal splines and their Fourier transforms.

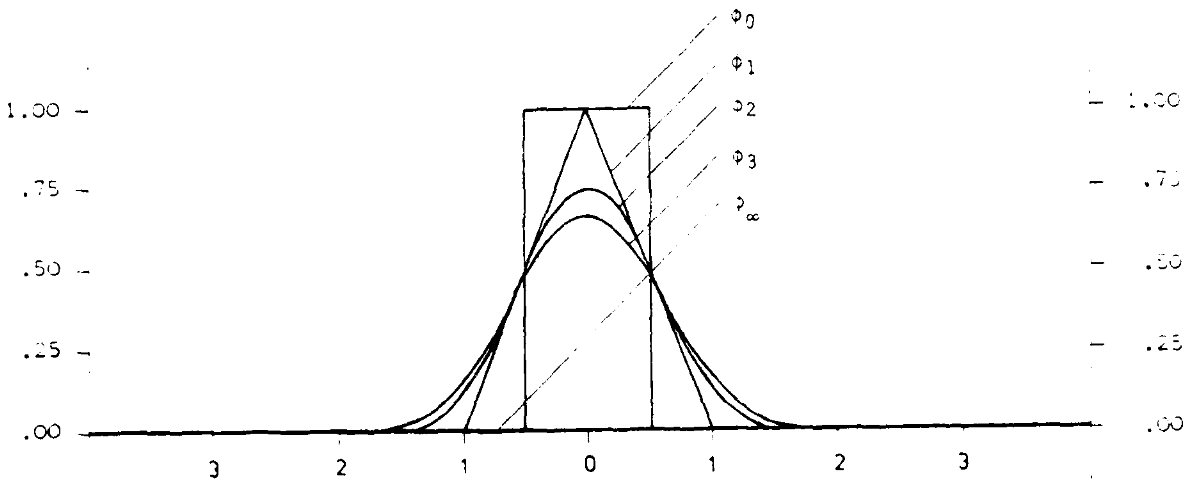

FIG. 1.9 Spline base functions ϕ_n of degree n

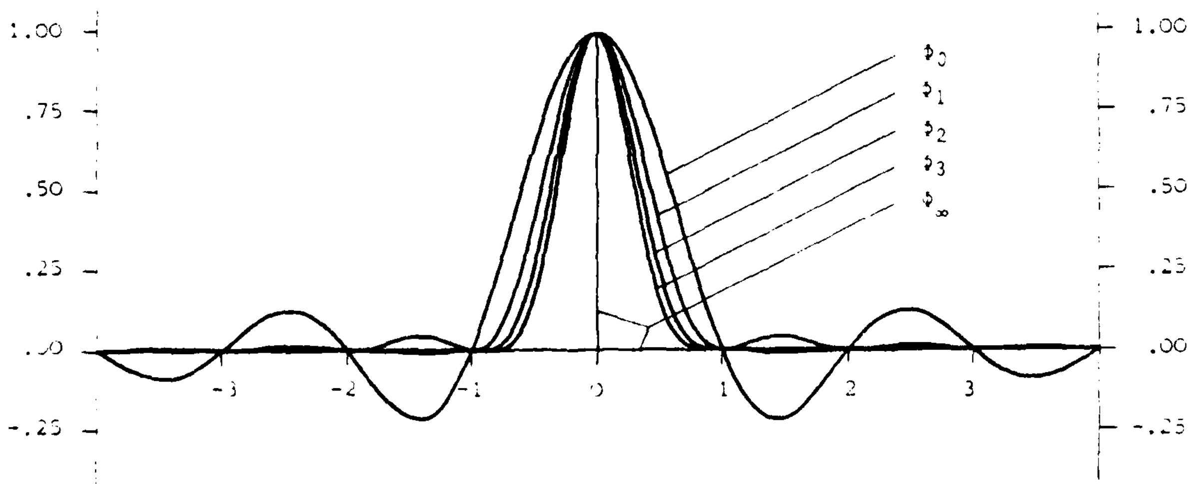

FIG. 1.10 Fourier transforms Φ_n of spline base functions of degree n

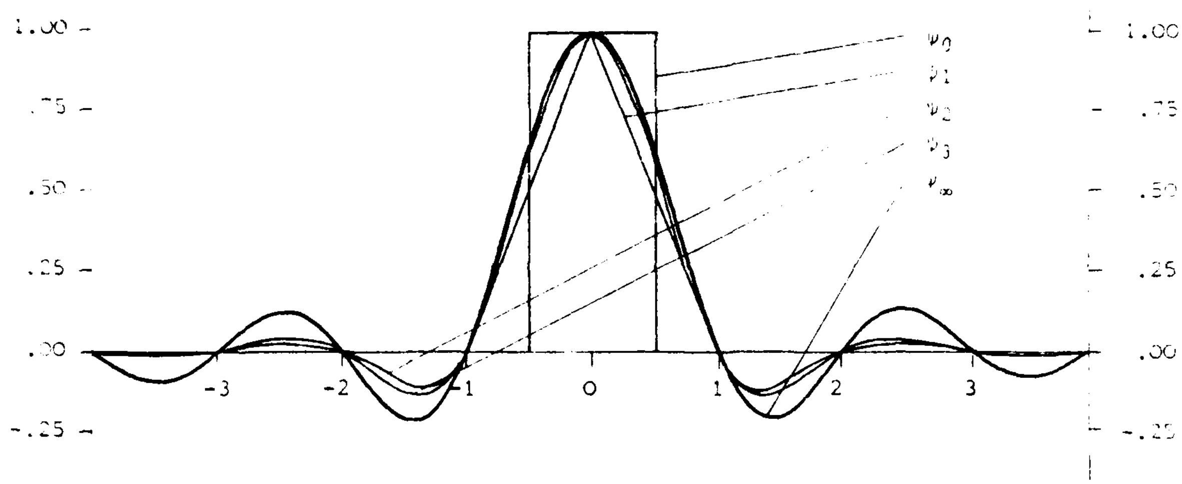

FIG. 1.11 Cardinal splines (= sampling splines) ψ_n of degree n

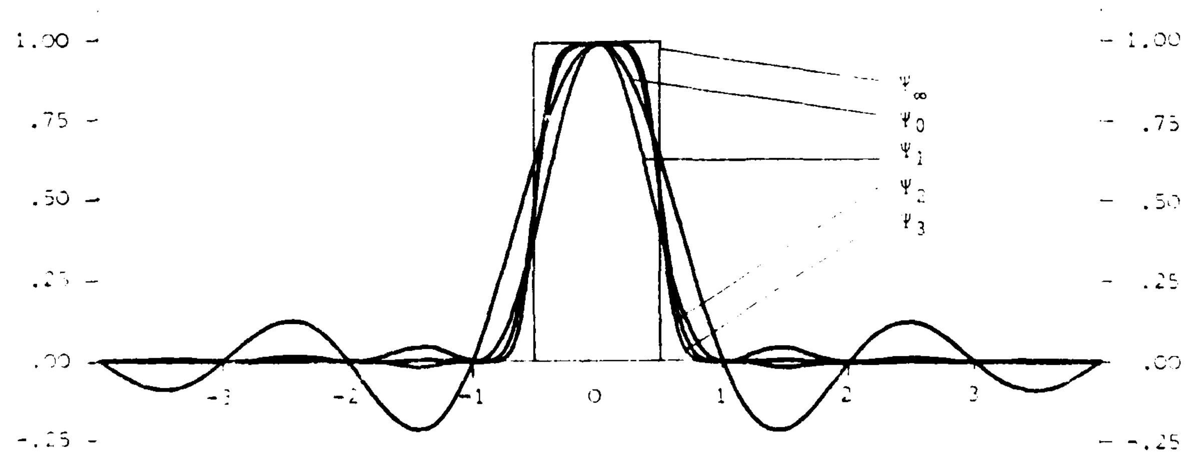

FIG. 1.12 Fourier transforms Ψ_n of cardinal splines of degree n

2. SPLINES AND THE GAUSSIAN FUNCTION

In the last chapter we intended to introduce the family of cardinal splines in a stepwise direct approach. We saw that there is a whole bunch of relations between these interpolation functions; section 1.5. focused on the goal of finding the cardinal spline of highest possible degree; it turned out to be the familiar $\sin\pi x/\pi x$ function.

In this chapter it should be investigated if there is any relation between splines and the Gaussian bell-shaped function. The basic starting point is again the cardinal spline base function of lowest possible degree, the zero degree unit-step function ϕ_0. We noted already during the discussion of higher than zero order spline base functions a couple of significant features:

a) the base function ϕ_0 has a constant height equal to 1 and a support length of 1 ;

b) the base function ϕ_n can be expressed as an n-fold convolution of the unit-step zero degree cardinal spline base function ϕ_0 expressed by equation (1.48);

c) with each increase by one degree the continuous differentiability increases by one order; ϕ_n is (n-1) - times continuously differentiable;

d) with each increase by one degree the support length increased by one unit; ϕ_n has a support of (n+1) units; at the same time the maximum value $\phi_n(0)$ becomes smaller; the functions flatten out with increasing degree;

e) the correlation length increases with increasing degree as can be seen from Fig. 1.9 ;

f) all base functions ϕ_n have a constant integral area equal to 1 ;

g) the sum of all function values at the cardinal numbers equals 1 for any ϕ_n .

We are making now a side-step and recall some facts known from probability theory: According to the central limit theorem, the density function of a sum of uniformly distributed independent random variables tends to a Gaussian function (Papoulis, 1965, p. 267). More specifically: if n independent random variables x_i have a constant density function which equals the step function ϕ_0 (equation (1.1)), it follows immediately that

$$E\{x\} = E\{\sum_i x_i\} = \sum_i E\{x_i\} = 0 \quad .$$

The variance of a single random variable is accordingly given by

$$E\{x_i^2\} = \int_{-1/2}^{1/2} x^2 dx = \frac{1}{12} \quad .$$

Since the random variables are independent as postulated above, the variance of the sum equals the sum of the variances,

$$E\{(\sum_i x_i)^2\} = \sum_i E\{x_i^2\} \quad ;$$

it follows that for the case discussed here $(i = 1, \ldots, n)$ the variance σ^2 is given by

$$\sigma^2 = \sum_{i=1}^{n} E\{x_i^2\} = \frac{n}{12} \tag{2.1}$$

The density function of the sum equals the (n-1)fold convolution of the individual densities ϕ_0 (which were assumed to be equal),

$$\gamma_n(x) = \underbrace{\phi_0(x) * \phi_0(x) * \ldots * \phi_0(x)}_{(n-1)\text{ times}} \quad . \tag{2.2}$$

The corresponding Gaussian density function is

$$\tilde{\gamma}(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}$$

and with (2.1)

$$\tilde{\gamma}(x) = \sqrt{\frac{6}{\pi n}} \; e^{-\frac{6x^2}{n}} \quad . \tag{2.3}$$

With γ_n and $\tilde{\gamma}_n$ the central limit theorem says that $\tilde{\gamma}_n$ approaches γ_n if n increases ,

$$\tilde{\gamma}_n \approx \gamma_n \tag{2.4a}$$

and under very general conditions,

$$\lim_{n \to \infty} \tilde{\gamma}_n = \gamma_n \quad . \tag{2.4b}$$

Let us now come back to the spline base functions. Comparing (1.48) with (2.2), we conclude that the cardinal spline base function of degree n, ϕ_n, equals the density function γ_{n+1} of $n+1$ independent random variables having equal density functions ϕ_0,

$$\phi_n = \gamma_{n+1} \quad . \tag{2.5}$$

According to (2.4a) and (2.4b), the Gaussian function $\tilde{\gamma}_{n+1}$, defined in (2.3), can be considered an approximation to ϕ_n,

$$\tilde{\gamma}_{n+1} \simeq \phi_n \tag{2.6a}$$

and in the limit

$$\lim_{n\to\infty} \tilde{\gamma}_{n+1} = \phi_\infty \quad . \tag{2.6b}$$

Therefore, the cardinal spline base function of degree n can be approximated by a Gaussian function,

$$\tilde{\phi}_n = \tilde{\phi}_n(x) = \sqrt{\frac{6}{\pi(n+1)}}\, e^{-\frac{6x^2}{n+1}} \tag{2.7}$$

The following graphs provide a comparison between exact cardinal spline base functions and their corresponding approximations by a Gaussian curve.

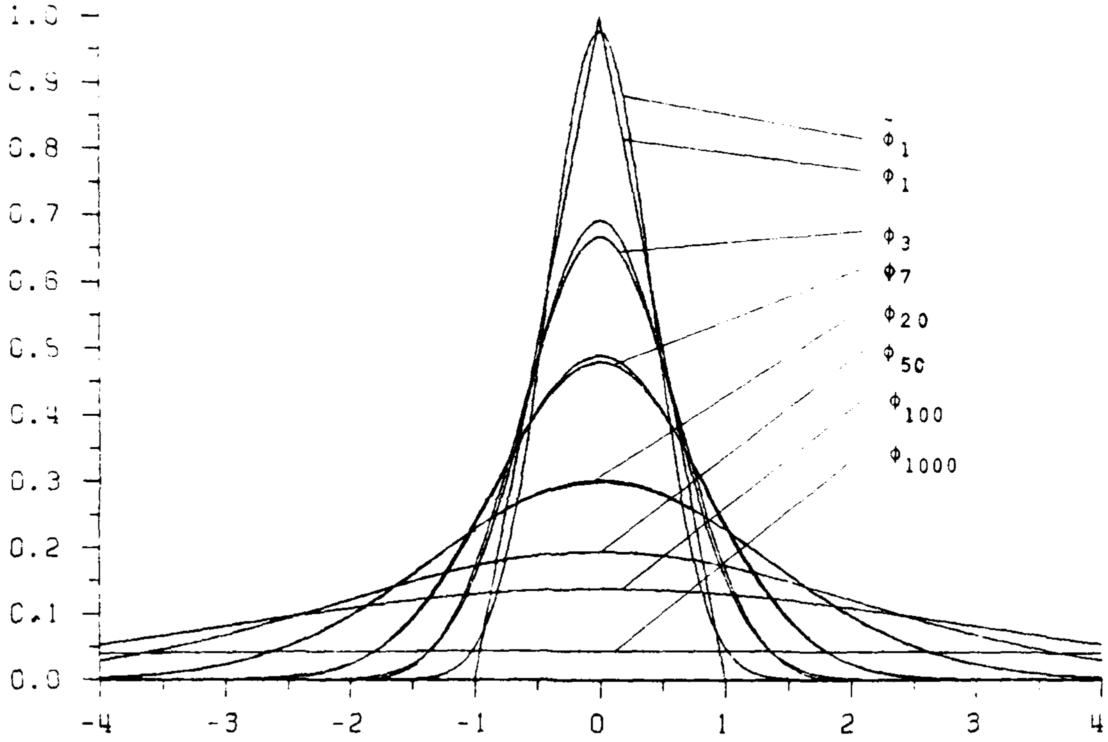

FIG. 2.1 Cardinal spline base functions and their approximations by a Gaussian curve.

The correlation length, which will be used later, can easily be shown to equal

$$h_n = \sqrt{\frac{(n+1)}{6} \ln 2} \tag{2.8}$$

for the Gaussian curve $\tilde{\phi}_n$ corresponding to ϕ_n . This formula enables us to obtain a good estimate for the correlation length of ϕ_n . We conclude:

The correlation length of the cardinal spline base function of degree n *is approximately proportional to* $\sqrt{n+1}$.

Another evident fact concerns the integral of $\tilde{\phi}_n$: we recall that $\tilde{\phi}_n = \gamma_{n+1}$ can be considered a density function; the integral is the distribution function and, consequently,

$$\int_{-\infty}^{\infty} \tilde{\phi}_n(x)\,dx = 1 \quad . \tag{2.9}$$

How does $\tilde{\phi}_n$ behave if n goes to infinity? We conclude from (2.7) that the maximum of $\tilde{\phi}_n$ (which is, of course, attained at the origin $x=0$) decreases like $1/\sqrt{n+1}$. Due to (2.8) the product

$$h_n \tilde{\phi}_n(0) = \sqrt{\frac{\ln 2}{\pi}} = \text{const.} \tag{2.10}$$

Let us denote the limit function by $\bar{\delta}(x)$,

$$\bar{\delta}(x) := \lim_{n\to\infty} \sqrt{\frac{6}{\pi(n+1)}}\; e^{-\frac{6x^2}{n+1}} \quad ; \tag{2.11}$$

Because of (2.8) and (2.10) $\bar{\delta}(x)$ is a kind of layer on the real line with infinitesimal function value and unlimited length but such that its volume equals 1 ,

$$\int_{-\infty}^{\infty} \bar{\delta}(x)\,dx = 1 \quad . \tag{2.12}$$

Since all $\tilde{\phi}_n$'s are frequency functions, the convolution between a function f and $\tilde{\phi}_n$ results in a smoothed function $\tilde{f}_n$,

$$\begin{aligned}\tilde{f}_n(x) &= f(x) * \tilde{\phi}_n(x) \\ &= \int_{-\infty}^{\infty} f(x')\tilde{\phi}_n(x' - x)\,dx' \quad .\end{aligned}$$

If n goes to infinity, it is obvious that

$$\int_{-\infty}^{\infty} f(x')\bar{\delta}(x' - x)\,dx' = \bar{f} \tag{2.13}$$

where $\bar{f}$ denotes the mean value of f . We conclude:

The result of a convolution between a function f and the distribution $\bar{\delta}$ equals the mean value of f .

This distribution will be used later.

2.3 SPECTRAL PROPERTIES OF GAUSSIAN SPLINE APPROXIMATION

The Fourier transform of $\tilde{\phi}_n$, the Gaussian curve approximating the cardinal spline base function of degree n (equation (2.7)) , is given by (Gradshteyn, 1971, p. 494, No. 3.896)

$$\tilde{\Phi}_n(\eta) = \int_{-\infty}^{\infty} \sqrt{\frac{6}{\pi(n+1)}}\, e^{-\frac{6x^2}{n+1}}\, e^{-i2\pi\eta x}\, dx \quad ,$$

$$\tilde{\Phi}_n(\eta) = e^{-(\pi\eta)^2 \frac{n+1}{6}} \quad . \tag{2.14}$$

Equation (2.14) can also be written as

$$\tilde{\Phi}_n(\eta) = \left(e^{-\frac{(\pi\eta)^2}{6}} \right)^{n+1} \tag{2.14'}$$

and shows once more the fact that $\tilde{\phi}_n$ resembles the n-fold convolution of Gaussian curves $\tilde{\phi}_0$,

$$\tilde{\Phi}_n = (\tilde{\Phi}_0)^{n+1}, \qquad \tilde{\Phi}_0(\eta) = e^{-\frac{(\pi\eta)^2}{6}} \quad . \tag{2.14''}$$

(Recall the corresponding spectrum of the base spline (subsection 1.5.1)),

$$\Phi_n = (\Phi_0)^{n+1} \quad \Phi_0(\eta) = \frac{\sin\pi\eta}{\pi\eta} \quad . \tag{2.15}$$

The spectra Φ_n and $\tilde{\Phi}_n$ are shown, for various degrees n , in Fig. 2.2, an upper bound for their absolute differences in Fig. 2.3.

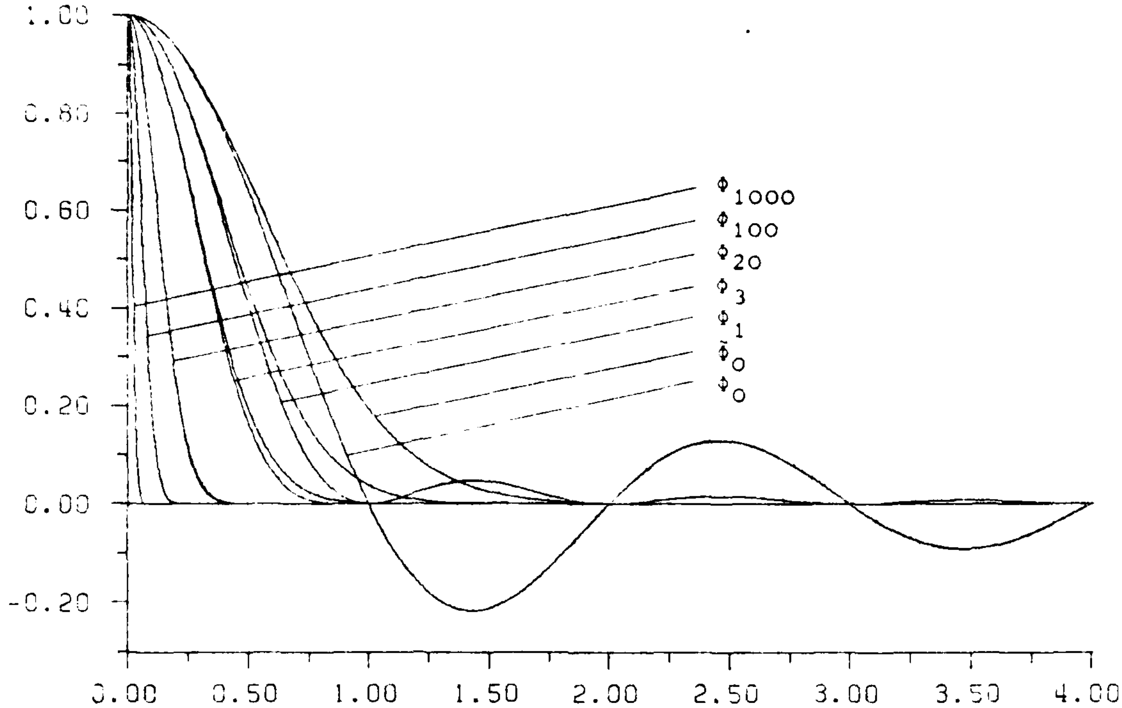

FIG. 2.2 Fourier transforms of cardinal spline base functions Φ_n and corresponding Gaussian functions $\tilde{\Phi}_n$

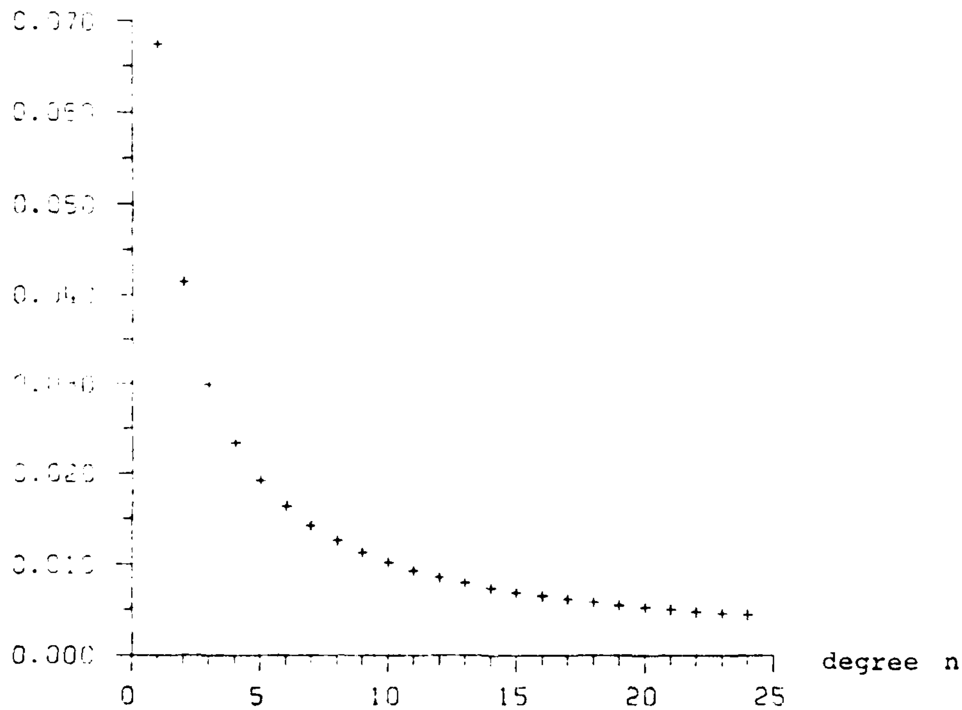

FIG. 2.3 Upper bound $\max_{\forall \eta} |\Phi_n(\eta)-\tilde{\Phi}_n(\eta)|$

From equations (2.14) and (2.15) as well as from Fig. 2.2 and 2.3 we conclude that both spectra Φ_n and $\breve{\Phi}_n$ become identical if n goes to infinity

$$\lim_{n\to\infty} \breve{\Phi}_n = \lim_{n\to\infty} \Phi_n \quad .$$

The limit spectrum equals 1 for the zero frequency and vanishes for all other frequencies. Note that Φ_∞ has no volume (mass), therefore it is <u>not</u> an impulse. We call this "function" $\bar{D}(\eta)$:

$$\bar{D}(\eta) \; := \lim_{n\to\infty} \Phi_n(\eta) \quad . \tag{2.16}$$

Recalling the definition of the distribution $\bar{\delta}(x)$ (equation (2.11)), it is obvious that $\bar{D}$ is the Fourier transform of $\bar{\delta}$,

$$\bar{D}(\eta) = \int_{-\infty}^{\infty} \bar{\delta}(x)\, e^{-i2\pi\eta x} dx \quad . \tag{2.17}$$

The inverse transform of $\bar{D}$ is the distribution $\bar{\delta}$,

$$\bar{\delta}(x) = \int_{-\infty}^{\infty} \bar{D}(\eta)\, e^{i2\pi\eta x} d\eta \quad . \tag{2.18}$$

Therefore, $\bar{\delta}(x)$ and $\bar{D}(\eta)$ can be considered as a Fourier transform pair.

The distribution $\bar{\delta}$ is very closely related to the Dirac distribution δ as summarized in the following table.

$\delta(x)$, $D(\eta)$	$\bar{\delta}(x)$, $\bar{D}(\eta)$
$\int_{-\varepsilon}^{\varepsilon} \delta(x)dx = 1$, $\varepsilon \to 0$	$\int_{-\infty}^{\infty} \bar{\delta}(x)dx = 1$
$\int_{-\infty}^{\infty} f(x)\delta(x - x')dx = f(x')$	$\int_{-\infty}^{\infty} f(x)\bar{\delta}(x - x')dx = \bar{f}$
$D(\eta) = \int_{-\infty}^{\infty} \delta(x)e^{-i2\pi\eta x}dx = 1$	$\bar{D}(\eta) = \int_{-\infty}^{\infty} \bar{\delta}(x)e^{-i2\pi\eta x}dx = \begin{cases} 1 \text{ at } \eta=0 \\ 0 \text{ else} \end{cases}$
$\delta(x) = \int_{-\infty}^{\infty} D(\eta)e^{i2\pi\eta x}d\eta$	$\bar{\delta}(x) = \int_{-\infty}^{\infty} \bar{D}(\eta)e^{-i2\pi\eta x}d\eta$

The mutual relation between both distributions becomes particularly clear by the relation

$$\int_{-\infty}^{\infty} \bar{D}(\eta)\delta(\eta)d\eta = \int_{-\infty}^{\infty} D(\eta)\bar{\delta}(\eta)d\eta = 1 \ , \qquad (2.19)$$

which can be verified using the above relations. Both distributions can be considered as extreme cases of kernels of a moving average integral operator M_η ,

$$M_\eta(\cdot) = \int_{-\infty}^{\infty} (\cdot)\delta_\eta dx \qquad (2.20a)$$

with δ_n, e.g., defined by

$$\delta_n(x) = \sqrt{\frac{6}{\pi n}}\; e^{-\frac{6x^2}{n}} \quad ; \tag{2.20b}$$

if n goes to zero, we obtain the Dirac distribution $\delta(x)$, if n goes to infinity, we obtain the $\bar{\delta}(x)$ - distribution; both limit operators M_0 and M_∞ degenerate into functionals; the evaluation functional M_0 (which "averages" the function within an infinitesimally small interval), and the averaging functional M_∞ (which averages the function on an infinitely large interval). It is due to its relatively unimportant properties that the distribution $\bar{\delta}(x)$ is not considered in the literature.

Let us now consider the Fourier transform of the Gaussian cardinal (sampling) function corresponding to the cardinal (sampling) function of degree n ,

$$\tilde{\psi}_n(x) = \sum_{-\infty}^{\infty} \tilde{\alpha}_k \, \tilde{\phi}_n(x - k) \quad . \tag{2.21}$$

According to subsection 1.5.1, the Fourier transform $\tilde{\Psi}_n(\eta)$ is given by

$$\tilde{\Psi}_n(\eta) = \Phi_n(\eta) \sum_{-\infty}^{\infty} \tilde{\alpha}_k \cos 2\pi k\eta \quad , \tag{2.22}$$

which, in turn, can simply be transformed to

$$\tilde{\Psi}_n(\eta) = \frac{\tilde{\Phi}_n(\eta)}{\tilde{a}_n(\eta)} \tag{2.23a}$$

with $\tilde{\Phi}_n(\eta)$ given by equation (2.14) and $\tilde{a}_n(\eta)$ by

$$\tilde{a}_n(\eta) = \sum_{-\infty}^{\infty} \tilde{\phi}_n(k)\cos 2\pi k\eta \quad . \tag{2.23b}$$

Explicitly written $\tilde{\Psi}_n(\eta)$ is as follows

$$\tilde{\Psi}_n(\eta) = \frac{e^{-(\pi\eta)^2 \frac{(n+1)}{6}}}{\sqrt{\frac{6}{\pi(n+1)}}\left\{1 + 2\sum_{1}^{\infty} e^{-\frac{6k^2}{n+1}} \cos 2\pi k\eta\right\}} \quad . \tag{2.24}$$

The following graph shows the maximum absolute differences between Ψ_n and $\tilde{\Psi}_n$ for different degrees n .

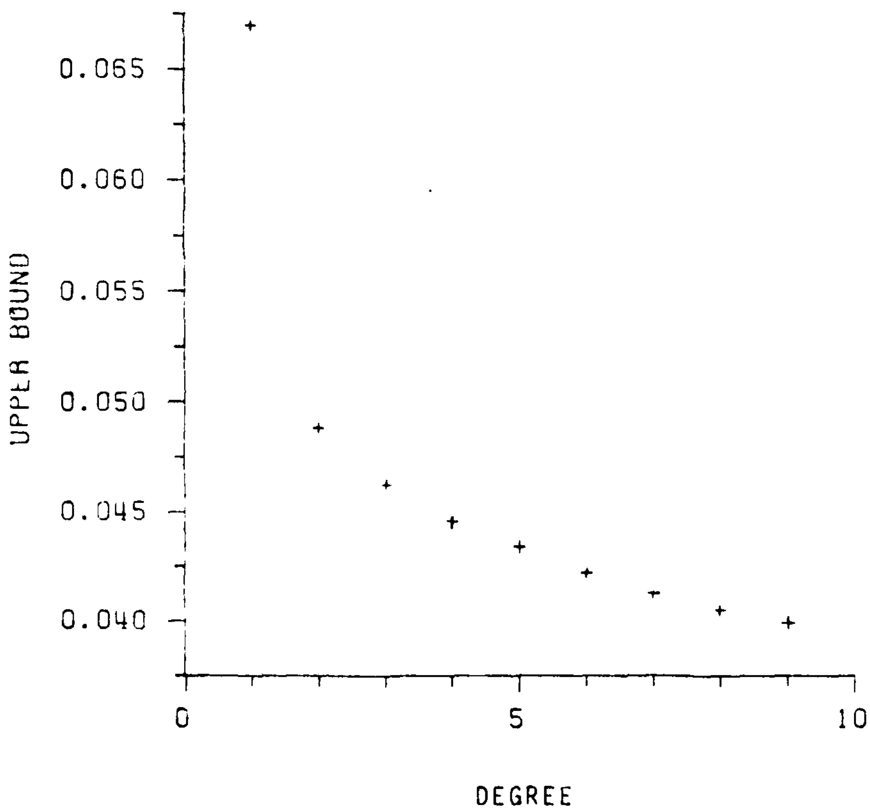

FIG. 2.4 Upper bound $\max\limits_{\forall\eta} |\Psi_n(\eta)-\tilde{\Psi}_n(\eta)|$

How does the limit spectrum $\tilde{\psi}_\infty(\eta)$ look like? Figure 2.5 demonstrates the behavior of $\tilde{\psi}_n$ for large n ($n = 10, 20, \ldots$); obviously $\tilde{\psi}_\infty$ is a rectangular pulse

$$\tilde{\psi}_\infty(\eta) = \begin{cases} 1 & \text{for } |\eta| < \frac{1}{2} \\ 0 & \text{else} \end{cases} \tag{2.25}$$

Comparing $\tilde{\psi}_\infty$ with ψ_∞ (equation (1.64)), we see that both are the Fourier transform of the function $\sin\pi x/\pi x$,

$$\tilde{\psi}_\infty(\eta) = \psi_\infty(\eta) = \int_{-\infty}^{\infty} \frac{\sin\pi x}{\pi x}\, e^{-i2\pi\eta x} dx \tag{2.26a}$$

and, consequently,

$$\tilde{\psi}_\infty(x) = \psi_\infty(x) = \frac{\sin\pi x}{\pi x} \quad . \tag{2.26b}$$

This result has already been obtained by a different approach in section 1.5.

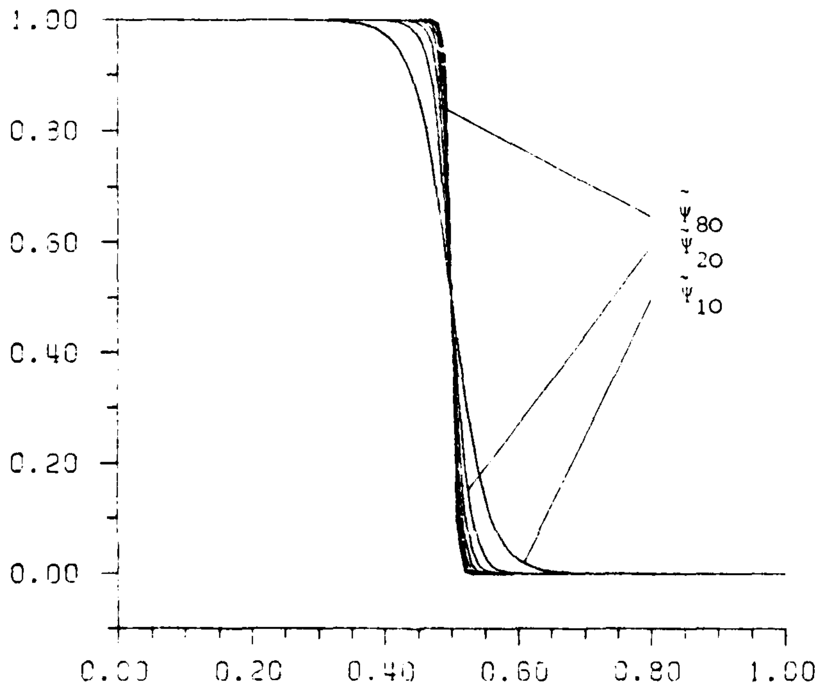

FIG. 2.5 Convergence of $\tilde{\Psi}_n(\eta)$, $n \to \infty$

2.2 DEVIATION OF GAUSSIAN CARDINAL BASE AND SAMPLING CURVES FROM THE CORRESPONDING SPLINES

It is of considerable interest to know a) how much the cardinal spline base functions differ from the corresponding Gaussian curves, and b) how much the cardinal sampling splines differ from the corresponding Gaussian ones. The first question has a simple answer. Generalizing equation (1.68) for real arguments, one arrives at the following expression for the cardinal spline base function of degree n,

$$\varphi_n(x) = \begin{cases} \dfrac{1}{n!} \displaystyle\sum_{j=0}^{\mathrm{int}\left(\frac{n+1}{2} + |x|\right)} (-1)^j \binom{n+1}{j} \left(\frac{n+1}{2} + |x| - j\right)^n & \text{for } 0 \le |x| < \dfrac{n+1}{2} \\ 0 & \text{else ;} \end{cases} \tag{2.27}$$

$\tilde{\varphi}_n(x)$ is given by (2.7),

$$\tilde{\varphi}_n(x) = \sqrt{\frac{6}{\pi(n+1)}}\; e^{-\frac{6x^2}{n+1}}$$

which makes the calculation of $\varphi_n - \tilde{\varphi}_n$ an easy task. The graph in Fig. 2.6 illustrates the convergence behavior of the deviation

$$\max_{\forall x} \frac{|\varphi_n(x) - \tilde{\varphi}_n(x)|}{\varphi_n(0)} \quad , \; n \to \infty \; .$$

Another estimate of an upper bound, which is, of course, more pessimistic, can be obtained via the spectra,

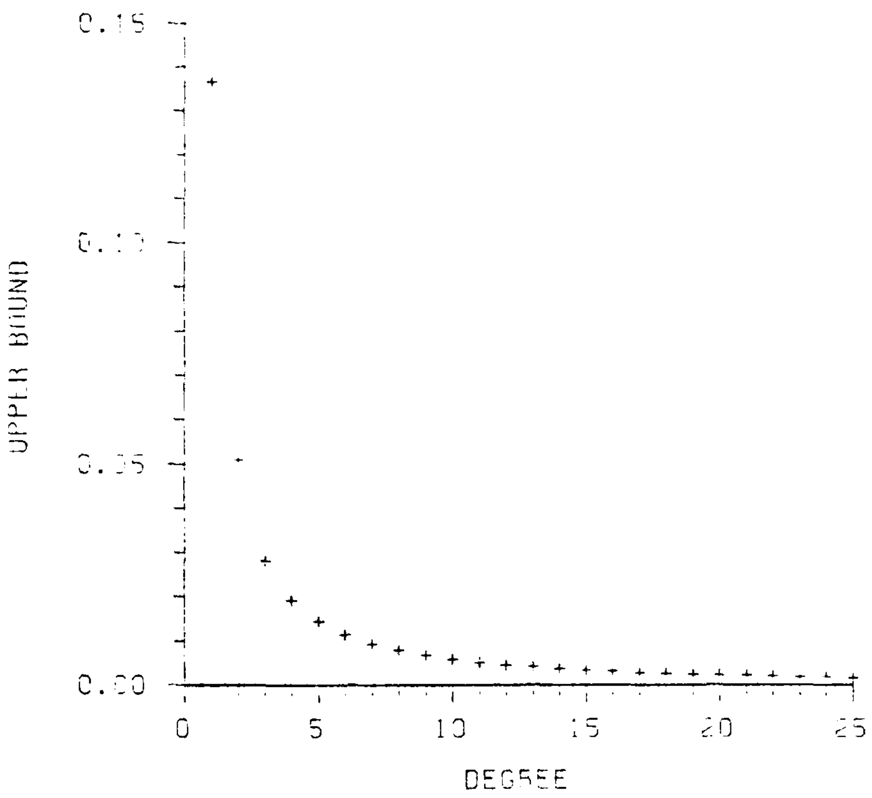

FIG. 2.6 Convergence of $\max\limits_{\forall x} \dfrac{|\phi_n(x)-\tilde{\phi}_n(x)|}{\phi_n(0)}$

$$
\begin{aligned}
|\phi_n(x) - \tilde{\phi}_n(x)| &= \left| \int_{-\infty}^{\infty} [\Phi_n(\eta) - \tilde{\Phi}_n(\eta)]\cos 2\pi\eta x d\eta \right| \\
&\leq 2 \int_{0}^{\infty} |\Phi_n(\eta) - \tilde{\Phi}_n(\eta)| d\eta \quad .
\end{aligned}
\tag{2.28}
$$

The second question of how much the cardinal sampling splines differ from the corresponding Gaussian curves, can virtually only

be answered by employing the spectra. The absolute difference, as above, can be represented in the following way

$$\begin{aligned}\psi_n(x) - \tilde{\psi}_n(x) &= \int_{-\infty}^{\infty} [\Psi_n(\eta) - \tilde{\Psi}_n(\eta)]\cos 2\pi\eta x d\eta \\ &\leq 2\int_0^{\infty} |\Psi_n(\eta) - \tilde{\Psi}_n(\eta)| d\eta\end{aligned} \tag{2.29}$$

with $\Psi_n(\eta) = \Phi_n(\eta)/\overline{a}(\eta)$ and $\tilde{\Psi}_n(\eta) = \tilde{\Phi}_n(\eta)/\tilde{a}(\eta)$; the explicit expressions are given by (1.71)' and (2.24). The graph in Fig. 2.7 demonstrates the convergence behavior of

$$\max_{\forall x} |\psi_n(x) - \tilde{\psi}_n(x)| \;, \quad n \to \infty \;;$$

(note that $\psi_n(0) = 1$ for all n ; $\tilde{\psi}_n(0)$, however, is less than 1 for $n > 1$.)

Fig. 2.8 shows cardinal sampling splines of various degrees and the corresponding Gaussian sampling curves.

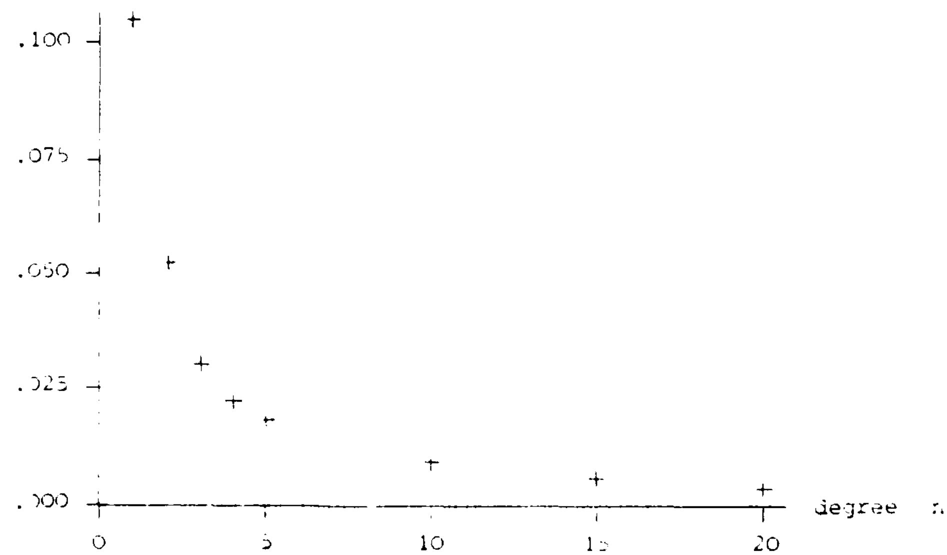

FIG. 2.7 Convergence of $\max_{\forall x} |\psi_n(x)-\tilde{\psi}_n(x)|$

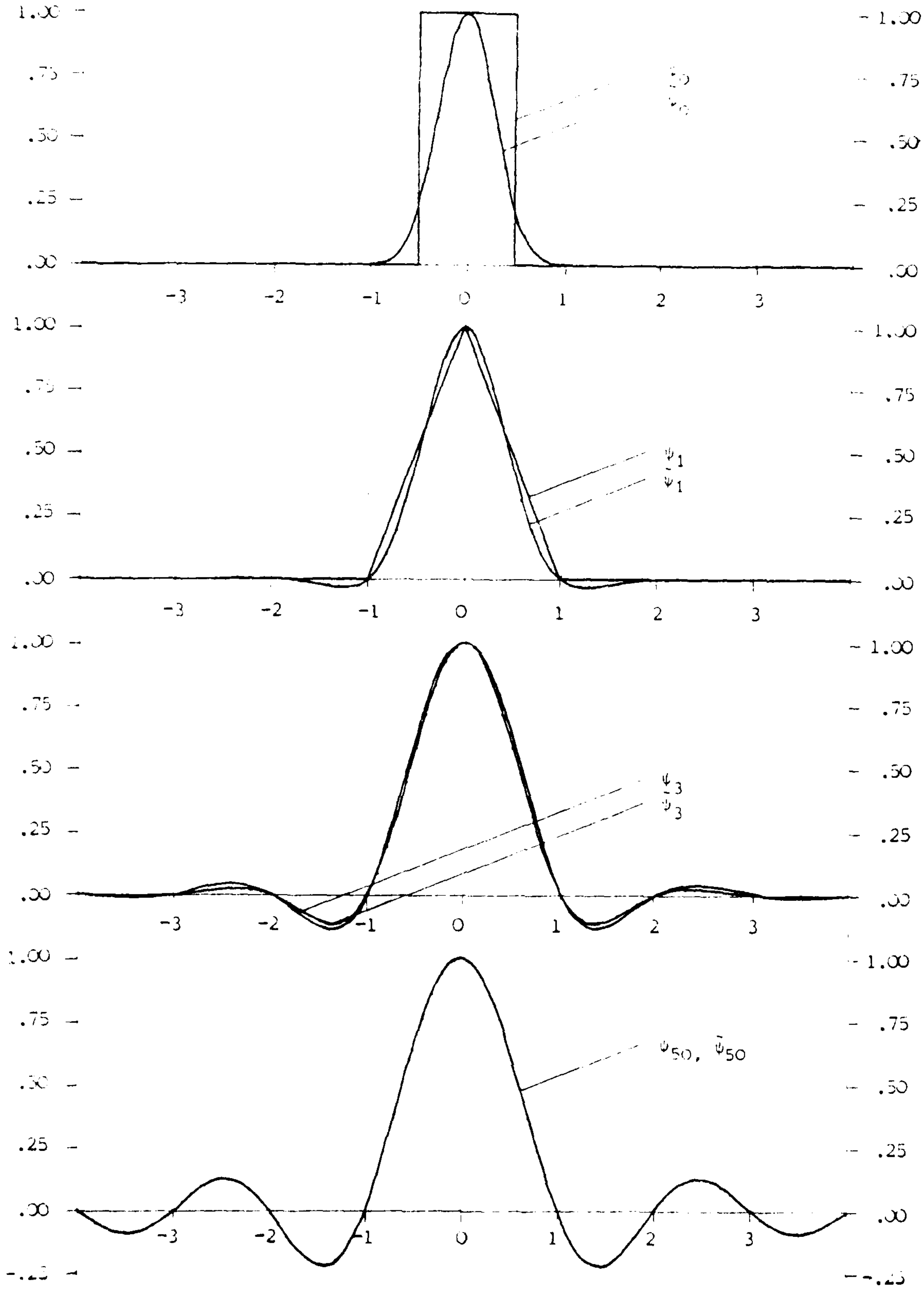

FIG. 2.8 Cardinal sampling splines and corresponding Gaussian sampling curves of various degrees

For prediction problems in geodesy the Gaussian covariance function is considered not as important as the covariance function of Hirvonen type which has the following general structure

$$\phi(x) = \left[1 + \left(\frac{x}{a}\right)^2\right]^{-\alpha} . \qquad (2.30)$$

Three values for α are of particular interest : α = 1/2, 1, 3/2; for a detailed discussion see (Moritz, 1976). The parameter a is related to the correlation length h through

$$h = a(2^{1/\alpha} - 1)^{1/2} .$$

For a cardinal data distribution the corresponding sampling function $\psi(x)$ is obtained in the same way as discribed for the Gaussian case. For the sake of completeness the sampling functions of Hirvonen type, dependent upon the correlation length h, are presented for α = 1/2, 1, and 3/2; the correlation length has been selected as h = 0.1, 1, and 10 in order to point out the essential features. Comparing Fig. 2.9 (Hirvonen) with Fig. 2.8 (Gauß) and taking into account equation 2.8, we conclude : to each Hirvonen sampling function an almost indistinguishable Gaussian sampling function exists; the correlation length of the Gaussian base function, however, has to be significantly smaller than the one of the corresponding Hirvonen base function. This statement is valid at least for the Hirvonen functions considered here (α = 1/2, 1, 3/2). This fact has a very strong impact on interpolation error estimation discussed in chapter 5 . With increasing correlation length h the Hirvonen sampling functions approach the $\sin\pi x/\pi x$ function; it shares this feature with the Gaussian sampling function.

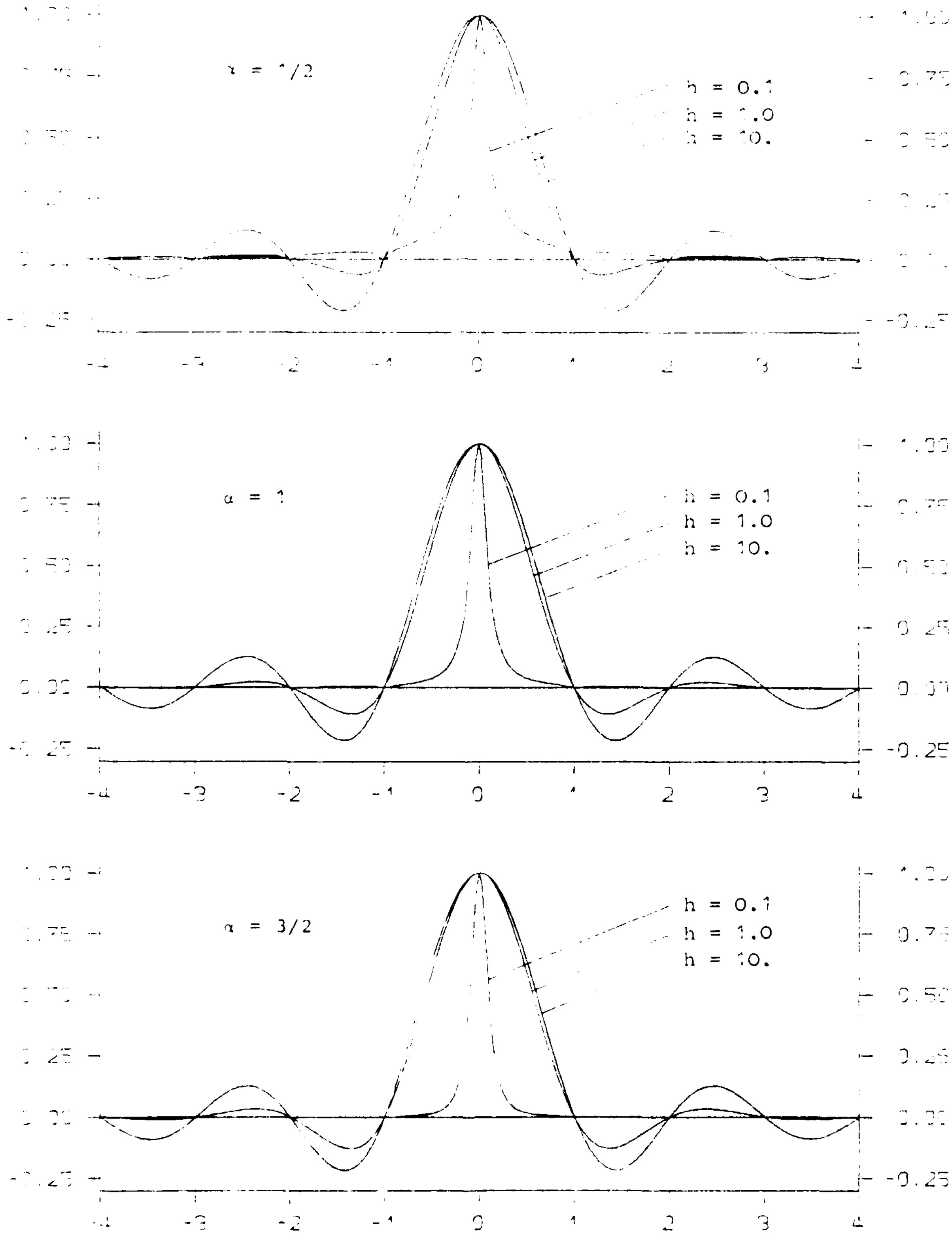

FIG. 2.9 Hirvonen cardinal sampling functions
(h ... correlation length)

3. SPLINE INTERPOLATION VERSUS LEAST-SQUARES INTERPOLATION

In the foregoing chapters it has been shown how splines of different degree are related to corresponding Gaussian curves. The link is provided by the central limit theorem or by the theory of linear systems in cascade (Papoulis, 1968, p. 50). All spline interpolations on a homogeneous set of regularly distributed data along the real line lead to the following form

$$\hat{f}_n(x) = \sum_{-\infty}^{\infty} f_k \psi_n(x - k) \quad ; \tag{3.1}$$

(for the sake of simplicity a data interval equal to unity has been assumed.) It is essential to point out that the function ψ is a sampling function whose support is, in general, the whole real line (if one ignores the countably infinite number of zeroes); only ψ_0 and ψ_1 have a limited support of 1 and 2, respectively. ψ_n is (n-1) times continuously differentiable and consists of piecewise polynomials of degree n. ψ_n is, apart from the cases n=0 and n=1, not directly given; it is rather the result of a linear transformation of originally defined base functions $\phi_n(x)$ such that

$$\psi_n(x) = \sum_{-\infty}^{\infty} \alpha_k \phi_n(x - k) \;.$$

The coefficients α_k can be determined from the sampling properties of ψ_n. This leads to the solution of the linear system

$$\phi_n(j - k)\alpha_k = \delta_{j,0} \;.$$

The system can be solved by means of frequency domain methods

and provides simple solutions for $\{\alpha_k\}$ if $n \leq 3$. Since the infinite matrix $\Phi := \{\phi_n(j - k)\}$ is of Toeplitz type, its inverse Φ^{-1} is also a Toeplitz matrix; consequently, the infinite solution vector $\alpha = \{\alpha_k\}$ represents any row or column of Φ^{-1}; Φ^{-1} is completely defined by α,

$$\alpha = \Phi^{-1}e \quad .$$

The interpolation function $\hat{f}_n(x)$ can also be formulated as a linear combination of the base functions $\phi_n(x)$,

$$\hat{f}_n(x) = \sum_{-\infty}^{\infty} \beta_k \phi_n(x - k) \quad . \tag{3.2}$$

According to the interpolation property there must hold

$$\hat{f}_n(j) = f_j = \sum_{-\infty}^{\infty} \beta_k \phi_n(j - k)$$

which yields the solution vector β

$$\beta = \Phi^{-1}f \quad ;$$

Φ^{-1} is given by α, therefore β is known (f is the data vector), and we obtain as solution to the interpolation problem

$$\hat{f}_n(x) = \phi_n(x - k)\{\phi_n(k - l)\}^{-1}f_l \quad .$$

(The summation over k and l is self-evident.) Briefly,

$$\hat{f}_p = \Phi_p^T \Phi^{-1} f \quad . \tag{3.3}$$

In order to be absolutely clear, let us once more explain, what

these symbols mean: f is the data vector, $\hat{f}_P$ the interpolated function value at the point P $(x=P)$, ϕ_P is the vector of translated base functions $\{\phi_n(P-k)\}$ evaluated at the point P $(x=P)$, ϕ is the matrix of transferred base functions $\phi_n(l-k)$ evaluated at the points $x = l$, $l, k = \ldots, -1, 0, 1, \ldots$.

Formally, the spline function interpolation solution (3.3) is identical with the least-squares interpolation solution (e.g. Moritz, 1980, p. 81)

$$\hat{f}_{c_P} = C_P^T C^{-1} f \quad . \tag{3.4}$$

If we consider the covariance function $C(x)$ a "base function", the relation becomes evident; all what has been said above about the base functions $\phi_n(x)$ can be literally copied for the covariance (base) function $C(x)$:

The interpolation function $\hat{f}_c(x)$ can be expressed as a linear combination of sampling functions

$$\hat{f}_c(x) = \sum_{-\infty}^{\infty} f_k \psi_c(x-k) \quad , \tag{3.5}$$

or by a direct linear combination of the covariance (base) functions

$$\hat{f}_c(x) = \sum_{-\infty}^{\infty} \beta_k C(x-k) \quad . \tag{3.6}$$

The interpolation property leads to the solution of the linear system

$$\hat{f}_c(j) = f_j = \sum_{-\infty}^{\infty} \beta_k C(j-k) \quad ,$$

which yields the solution

$$\beta = C^{-1} f \quad ;$$

the interpolation function $\hat{f}_c(x)$ is finally given by

$$\hat{f}_c(x) = C(x - k)\{C(k - l)\}^{-1} f_l \quad .$$

Note that the least-squares sampling function $\phi_c(x)$ is expressed as a linear combination of the covariance functions; the coefficients of the linear combination are the elements of the inverse covariance matrix C^{-1} ,

$$\phi_c(x - k) = C^{-1}_{kj} C(x - j) \quad .$$

This can immediately be verified by equating (3.5) and (3.6), observing $\beta = C^{-1} f$.

The two solutions share many common features (also the covariance matrix is of Toeplitz type, etc.); there is one very essential difference, however, between both solution, which is due to the structure of the base function : the covariance (base) function has an unlimited support, therefore, the covariance matrix is a full matrix; the spline base functions have all a limited support, the corresponding spline matrices are, therefore, band matrices with a bandwidth N depending on the degree n of the spline,

$$N = 2 \cdot \mathrm{int}\left(\frac{n}{2}\right) + 1 \quad . \qquad (3.7)$$

Band-structured matrices can be inverted much faster than full ones using a modified Cholesky decomposition algorithm; needless to say, the speed depends on the actual dimension of the matrix and on its bandwidth : if M denotes the dimension of the matrix

and n the degree of the spline, the inversion time is proportional to $\mathrm{int}(\frac{n}{2})^2 \cdot M$ as compared to M^3 for a full matrix (Forsythe and Moler, 1967); with other words, a band-matrix inversion is on the order of $M/\mathrm{int}(\frac{n}{2})^2$ faster than the inversion of a full matrix. This fact, among others, makes spline interpolation particularly attractive.

Let us put a very crucial question: Can a spline interpolation replace a least-squares interpolation?

In chapter 2 it was demonstrated how splines of increasing degree approach the Gaussian curve. If, therefore, the covariance function is of Gaussian type, and if a spline of sufficiently high degree is chosen, we can expect that least-squares interpolation and spline interpolation give practically the same result; this should be true not only for the interpolated value, but also for the predicted interpolation error. The degree n of the spline depends only on the correlation length of the corresponding Gaussian covariance function through equation (2.8) ,

$$n = \mathrm{int}\left(\frac{6}{\ln 2} h^2\right) - 1 \quad . \tag{3.8}$$

(In this context, h refers to a data spacing equal to unity.) Since the correlation distance of spline base functions increases only with the square root of the degree n , in general, a very high degree spline has to be chosen; as a consequence, the bandwidth of the spline-approximated covariance matrix will increase: according to equation (3.7) the bandwidth increases linearly with the degree n and, considering (3.8) , quadratically with the normalized correlation length of the covariance function.

E.g. : Assume 1000 data (M = 1000) ; a full inversion on a very fast AMDAHL 470 system would require some 20 minutes of CPU-time ; the following estimates are obtained for corresponding spline solutions:

h	n	CPU (sec)
1	7	0.01
2	33	0.3
3	77	1.6
4	137	5.3
5	215	12.8

The band matrix becomes a full one if

$$h > \sqrt{\frac{M+1}{6} \ln 2} \quad ,$$

which corresponds to $h \doteq 10.8$ in our example. If the degree n increases, the calculation of a spline function value according to (1.68) becomes expensive. This should not cause any problems, however, since in practical applications one will generate a high density table for the base function once for all and find an actual function value by a simple table interpolation function; this is also a common practice in applications of least-squares collocation, where covariances are usually tabulated; an inexpensive interpolation algorithm is described in (Sünkel, 1978, 1979) .

So far we have assumed uniform data distribution which is hardly ever available in practical applications. Spline as much as least-squares interpolation, however, is by no means restricted to regular distributions. For irregularly distributed data the basic equations (3.3) and (3.4), respectively, still hold, although the basic features of the spline- and corresponding covariance matrices disappear : the matrices are no longer of Toeplitz type ($C_{jk} \neq C(j-k)$, in general); therefore, the spline matrix is no longer a band-matrix; it is, however, still a sparse matrix with the number of zeroes approximately equal to that in the cor-

responding band-matrix (the correspondence referring to an average data spacing).

If a least-squares interpolation is replaced by a corresponding spline interpolation, it is required that the spline base function (which replaces the covariance function) be positive semi-definite which, in terms of the spectrum, means that the spectrum of the introduced base function has to be non-negative. We know from chapter 1 that the spectrum of ϕ_n is given by

$$\left(\frac{\sin\pi\eta}{\pi\eta}\right)^{n+1} ;$$

since $\sin\pi\eta/\pi\eta$ is positive and negative, we need an even power in order to keep the spectrum non-negative; an even power of $\sin\pi\eta/\pi\eta$ corresponds to an odd degree spline base function. Therefore, we conclude that only odd degree spline iterpolations can be used to replace a corresponding least-squares interpolation.

An interesting question arises in connection with the above described relation between spline and least-squares interpolation : It can be shown that the spline interpolant of degree $2n-1$ minimizes the quadratic functional

$$\int_{-\infty}^{\infty} |\hat{f}^{(n)}_{(x)}|^2 dx = \text{minimum} \qquad (3.9)$$

with $f^{(n)}_{(x)}$ denoting the n'th order derivative of $f(x)$ (Ahlberg et al., 1967) . Isn't it possible to relate this norm minimization problem of the spline to the norm minimization problem of least-squares interpolation,

$$f^T C^{-1} f = \text{minimum} \quad ? \qquad (3.10)$$

Let us express $\hat{f}$ in equation (3.9) by the spline given in

equation (3.3),

$$\hat{f}(x) = \phi^T(x)\phi^{-1}f \quad ,$$

$$\hat{f}''(x) = \phi''^T(x)\phi^{-1}f \quad ; \tag{3.11}$$

with (3.11) the minimum condition (3.9) is transformed into

$$\int_{-\infty}^{\infty} [\hat{f}^{(n)}(x)]^2 dx = f_j \phi_{jk}^{-1} \left[\int_{-\infty}^{\infty} \phi^{(n)}(x - k)\phi^{(n)}(x - l)dx\right]\phi_{lm}^{-1} f_m \quad . \tag{3.12}$$

Here we have explicitly expressed the dependence of ϕ on the difference $x-k$ and $x-l$, respectively. With (3.12) the minimum condition (3.9) can be transformed such that its dependence on the data is obvious,

$$f^T\phi^{-1}K\phi^{-1}f = \text{minimum} \quad ,$$

$$K_{kl} := \int_{-\infty}^{\infty} \phi^{(n)}(x - k)\phi^{(n)}(x - l)dx \quad . \tag{3.12'}$$

It is clear that (3.12)' would reduce to the least-squares interpolation condition of the form (3.10) if the matrix K would equal ϕ. Does it?

K represents obviously a convolution; therefore, the elements of K depend only on the difference $k-l$,

$$K_{kl} = K(k - l) \quad ;$$

consequently, K is a Toeplitz matrix. The function $\phi^{(n)}(x)$ is the n-fold derivative of the spline base function $\phi_{2n-1}(x)$ –

in the notations above we have consistently suppressed the degree $2n-1$ – which has a support of length equal to $2n$; since $\phi(|x| + \epsilon) < \phi(|x|)$, $\epsilon > 0$ (1) within the support $|x| < n$, it follows that the support of the derivative $\phi^{(n)}$ is also equal to $2n$. The convolution $\phi^{(n)} * \phi^{(n)}$ has twice the support of ϕ ; consequently K cannot equal ϕ and the answer of the above question is in the negative.

Furthermore, utilizing (1.68) , it can be shown that the sum of each row (or column) of K equals zero, whereas the sum of each row (or column) of ϕ equals 1 . Such kind of matrices are known as "stochastic matrices" (Zurmühl, 1964, pp. 221-224). Such kind of matrices are known to have a simple eigenvalue which equals the row-sum; the corresponding inverses have an eigenvalue which equals the reciprocal value of the row-sum. The product of two stochastic matrices has an eigenvalue which equals the product of their row-sums. Therefore, ϕ and ϕ^{-1} have an eigenvalue equal to 1 , K has a vanishing eigenvalue and, consequently, the product $\phi^{-1}K\phi^{-1}$ has also a vanishing eigenvalue. With other words, the "metric" $\phi^{-1}K\phi^{-1}$ is singular. As a consequence, there are data vectors f which are not identically zero and have still a vanishing norm in the space defined by the above metric. Such norms are called "semi-norms"; they are characteristic for spline functions. E.G.: The norm minimization problem (3.9) with $n=1$ (linear spline) is "blind" with respect to a constant function; the norm minimization problem with $n=2$ (cubic spline) is "blind" with respect to a constant function and with respect to a linear function with constant derivative; in general, the norm minimization problem (3.9) for the spline of degree $2n-1$ is "blind" with respect to all functions $g(x)$ and all linear combinations thereof, having constant derivatives $D^j g(x)$, $j = 0, \ldots, n-1$; these are obviously polynomials of degree $n-1$. This "blindness", however, does not have any consequences in the case studied here, as will

(1) This is a consequence of ϕ being a density function.

be argued in the sequel:

The minimization problem (3.9) is admissible in a space of functions having square integrable n'th order derivatives, called L_2^n ; then it follows that the data vector has to be an element of l_2^n , the space of square integrable n'th order divided differences. (For an explanation of divided differences see, e.g. Davis, 1975, pp. 40, 64 ff..) It can be shown that under these presumptions and the interpolation condition, there exists a unique solution to the minimization problem (3.9) such that the interpolating function $\hat{f}(x)$ is a spline of degree $2n-1$ (Schoenberg, 1973, p. 59ff.)

The norm for the least-squares solution differs considerably from the norm (3.9) : since the covariance matrix is positive definite, the norm (3.10) vanishes if and only if the data vector vanishes identically. In chapter 2 it was shown that the spline solution approaches the least-squares solution if the correlation length (and, accordingly, the degree of the spline) increases. How can two thus different minimization problems lead to practically the same result?

The key is obviously given by the spline norm on the one hand and the relation between splines and the Gaussian function on the other hand : the spline of degree $2n-1$ minimizes the integral of its squared n'th derivatives and interpolates the data. As stated before, this seminorm minimization is blind with respect to a polynomial of degree $n-1$; due to the interpolation condition, however, this polynomial is, so to say, "smuggled in"; the higher the spline's degree, the less important is the corresponding norm minimization and the more pronounced is the (single) polynomial's influence on the behavior of the interpolation function. (It should be evident that the minimization of the squared, e.g. 5'th order, derivatives contributes only very little to the shape of the function.) In the limit, if the spline's degree goes to infinity, its corresponding sampling function equals the interpolation function

$$\hat{f}_{\infty}(x) = \sum_{-\infty}^{\infty} f_k \frac{\sin\pi\eta(x-k)}{\pi(x-k)} \tag{3.13}$$

which has been shown in section 1.5. Therefore, if our conclusions concerning the polynomial-like behavior of high degree splines are valid, the above spline interpolation of infinite degree should behave like a single polynomial of infinite degree interpolating the cardinal data vector. More specifically, the polynomial interpolation needs to be expressible as a linear combination

$$\hat{f}_p(x) = \sum_{-\infty}^{\infty} f_k \gamma_{\infty}(x - k) \tag{3.14a}$$

with $\gamma_{\infty}(x)$ being a sampling polynomial,

$$\gamma_{\infty}(x) = \begin{cases} 1 & \text{at} \quad x = 0 \\ 0 & \text{at} \quad x = k \neq 0 \\ \neq 0 & \text{else} \end{cases} \quad . \tag{3.14b}$$

This sampling polynomial is a Lagrange polynomial of infinite degree given by (e.g. Moritz, 1978, p. 3) ,

$$\gamma_{\infty}(x - k) = \frac{\ldots\ (x-k+2)(x-k+1)(x-k-1)(x-k-2)\ \ldots}{\ldots\ 2 \cdot 1 \cdot (-1) \cdot (-2)\ \ldots} \ . \tag{3.15}$$

Whittaker (1935, p. 62ff.) verified that, in fact, this Lagrange sampling polynomial equals the $\sin\pi\eta/\pi\eta$ - sampling function,

$$\gamma_{\infty}(x) = \frac{\sin\pi x}{\pi x} \ . \tag{3.16}$$

This remarkable relation provides us with the perfect proof that high degree splines tend to behave like single polynomials; the spline of infinite degree is, in fact, a single polynomial of infinite degree – the minimization of

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} [f^{(n)}(x)]^2 \, dx = \text{minimum}$$

looses its meaning completely.

Let us return to the above question. It was shown in chapter 2 that splines approach the Gaussian curve with increasing correlation length and correspondingly increasing degree. Since both interpolations, the spline and the least-squares interpolation, have the same structure (equations (3.3) and (3.4), we conclude that the norm minimization (3.10) of the least-squares solution becomes meaningless if the correlation length increases in very much the same way as the norm minimization (3.9) of the spline solution becomes meaningless with increasing degree. Therefore, both solutions have to behave in the same way in this limit case – like a polynomial interpolation.

4. BJERHAMMAR INTERPOLATION

Both the least-squares interpolation and spline interpolations share a common feature: the determination of the interpolation function requires an inversion of a matrix; its size equals the number of data for the case of least-squares interpolation, it reduces to a band matrix for the case of spline interpolations with a bandwidth depending on the degree of the spline (or on the correlation length). With other words, the sampling functions are obtained through an inversion process, a usually expensive undertaking. Is there a method which avoids this inversion -- can't the sampling functions be replaced by an a priori weight function? This question has been considered by various authors like Bjerhammar (1973) or Shepard (1964). We will consider in detail only the method proposed by Bjerhammar because it was and is still being considered a particularly useful tool for data interpolation by members of the geodetic community.

Bjerhammar proposes to define the interpolation function as a weighted average of the data with weights depending on the distances between the prediction point and the data point,

$$\hat{f}(x) = \sum_{m=1}^{M} f_m \beta(|x-x_m|) \quad . \tag{4.1}$$

In this context β denotes the weight function; it replaces the sampling function used before. In the case studied here, an infinite data set is given, defined at the cardinal numbers,

$$\hat{f}(x) = \sum_{-\infty}^{\infty} f_m \beta(|x-m|) \quad . \qquad (4.1)'$$

Since $\hat{f}(x)$ interpolates the data sequence $\{f_m\}$, it follows that $\beta(x)$ needs to fulfil the conditions of a sampling (weight) function,

$$\beta(j) = \delta_{j0} \quad ,$$
$$\sum_{m=-\infty}^{\infty} \beta(|x-m|) = 1 \qquad \forall\ x \quad . \tag{4.2}$$

The weight functions

$$\beta_p(x) := \frac{x^{-(p+1)}}{\sum_{-\infty}^{\infty} (x-k)^{-(p+1)}}$$

are proposed by Bjerhammar[1]; the positive quantity p will be denoted power of prediction.

Particularly simple is the Bjerhammar sampling function if $p = 1$:

$$\beta_1(x) = \frac{1}{x^2 \sum_{-\infty}^{\infty} (x-k)^{-2}} \quad . \tag{4.3}$$

The infinite sum in the denominator can be expressed by a closed expression (Gradshteyn, 1971, p. 50, No. 1.422),

$$\sum_{-\infty}^{\infty} \frac{1}{(x-k)^2} = \left(\frac{\pi}{\sin\pi x}\right)^2 \quad ;$$

(1) ... Here the smoothing quantity has been neglected.

therefore, $\beta_1(x)$ is given by

$$\beta_1(x) = \left(\frac{\sin \pi x}{\pi x}\right)^2 \quad , \qquad (4.3)'$$

an expression which is already familiar to us : $\beta_1(x)$ is nothing but the Fourier transform of the linear interpolation sampling function, a very astonishing result (cf. equation (1.17)) ,

$$\beta_1(x) = \Psi_1(x) \quad . \qquad (4.4)$$

(It has been shown in section 1.2 that $\Psi_1 \equiv \Phi_1$.)

The corresponding interpolation function is simply given by

$$\hat{f}(x) = \sum_{-\infty}^{\infty} f_m \left[\frac{\sin \pi(x-m)}{\pi(x-m)}\right]^2 \quad .$$

Since the sampling function β_1 equals the Fourier transform of the spline sampling function Ψ_1 , it follows that the spectrum B_1 of β_1 equals the spline sampling function ψ_1 (confer section 1.2),

$$B_1(\eta) = \int_{-\infty}^{\infty} \beta_1(x) e^{-i2\pi\eta x} dx \qquad (4.5)$$

$$= \begin{cases} 1 - |\eta| & \text{for } |\eta| \le 1 \\ 0 & \text{else} \end{cases} \quad ,$$

$$B_1(\eta) = \psi_1(\eta) \quad . \qquad (4.5)'$$

We conclude:
The Bjerhammar sampling function of power 1 equals the Fourier transform of the spline sampling function of degree 1 and vice versa.

Let us now consider the Bjerhammar sampling function of power 3 :

$$\beta_3(x) = \frac{1}{x^4 \sum_{-\infty}^{\infty} (x-k)^{-4}} \quad . \tag{4.6}$$

The infinite sum in the denominator can be expressed by

$$\sum_{-\infty}^{\infty} \frac{1}{(x-k)^4} = \frac{1 + 2\cos^2 \pi x}{3} \left(\frac{\pi}{\sin \pi x}\right)^4$$

(Hansen, 1975, p. 110, No. 6.1.134); using the relation $2\cos^2 \pi x = 1 + \cos 2\pi x$, β_3 reduces to

$$\beta_3(x) = \frac{3}{2 + \cos 2\pi x} \left(\frac{\sin \pi x}{\pi x}\right)^4 \quad . \tag{4.6'}$$

This is exactly the Fourier transform Ψ_3 of the cubic sampling spline ψ_3 ,

$$\beta_3(x) = \Psi_3(x) \quad . \tag{4.7}$$

The corresponding interpolation function is given by

$$\hat{f}(x) = \sum_{-\infty}^{\infty} f_m \frac{3}{2 + \cos 2\pi(x-m)} \left(\frac{\sin\pi(x-m)}{\pi(x-m)}\right)^4 .$$

Arguing as before, the Fourier transform B_3 of the sampling function β_3 equals the spline sampling function ψ_3,

$$B_3(n) = \psi_3(n) . \qquad (4.8)$$

We conclude:
The Bjerhammar sampling function of power 3 equals the Fourier transform of the spline sampling function of degree 3 and vice versa.

How does the Bjerhammar interpolation behave if the power of prediction p increases beyond any limit?

The limit sampling function

$$\lim_{p\to\infty} \beta_p(x) = \lim_{p\to\infty} \frac{x^{p+1}}{\sum_{-\infty}^{\infty} \frac{1}{\left(x-k\right)^{p+1}}}$$

can be simplified to

$$\lim_{p\to\infty} \beta_p(x) = \lim_{p\to\infty} \frac{1}{\sum_{-\infty}^{\infty} \frac{1}{\left(1-\frac{k}{x}\right)^{p+1}}}$$

$$= \lim_{p\to\infty} \left[\ldots + \frac{1}{\left(1+\frac{1}{x}\right)^{p+1}} + 1 + \frac{1}{\left(1-\frac{1}{x}\right)^{p+1}} + \ldots \right]^{-1} .$$

It is obvious that $B_\infty(x)$ equals 1 for $|x| < 0.5$, assumes the value 0.5 for $|x| = 0.5$, and vanishes for $|x| > 0.5$,

$$B_\infty(x) = \begin{cases} 1 & \text{for} \quad |x| < \frac{1}{2} \\ \frac{1}{2} & \text{for} \quad |x| = \frac{1}{2} \\ 0 & \text{else} \end{cases} . \qquad (4.9)$$

This is exactly the step function introduced at the beginning of chapter 1 , it is at the same time the Fourier transform of the sampling spline of infinite degree. Therefore, we conclude:

The behavior of the Bjerhammar interpolation function approaches that of a step function "interpolation" if the power of prediction p increases.

Fig. 1.12 shows Bjerhammar sampling functions (odd degree) of various degrees. Fig. 4.1 is an example of Bjerhammar interpolation with power of prediction $p = 3$; the data are located at the cardinal numbers; the step-like behavior of the interpolation function is obvious.

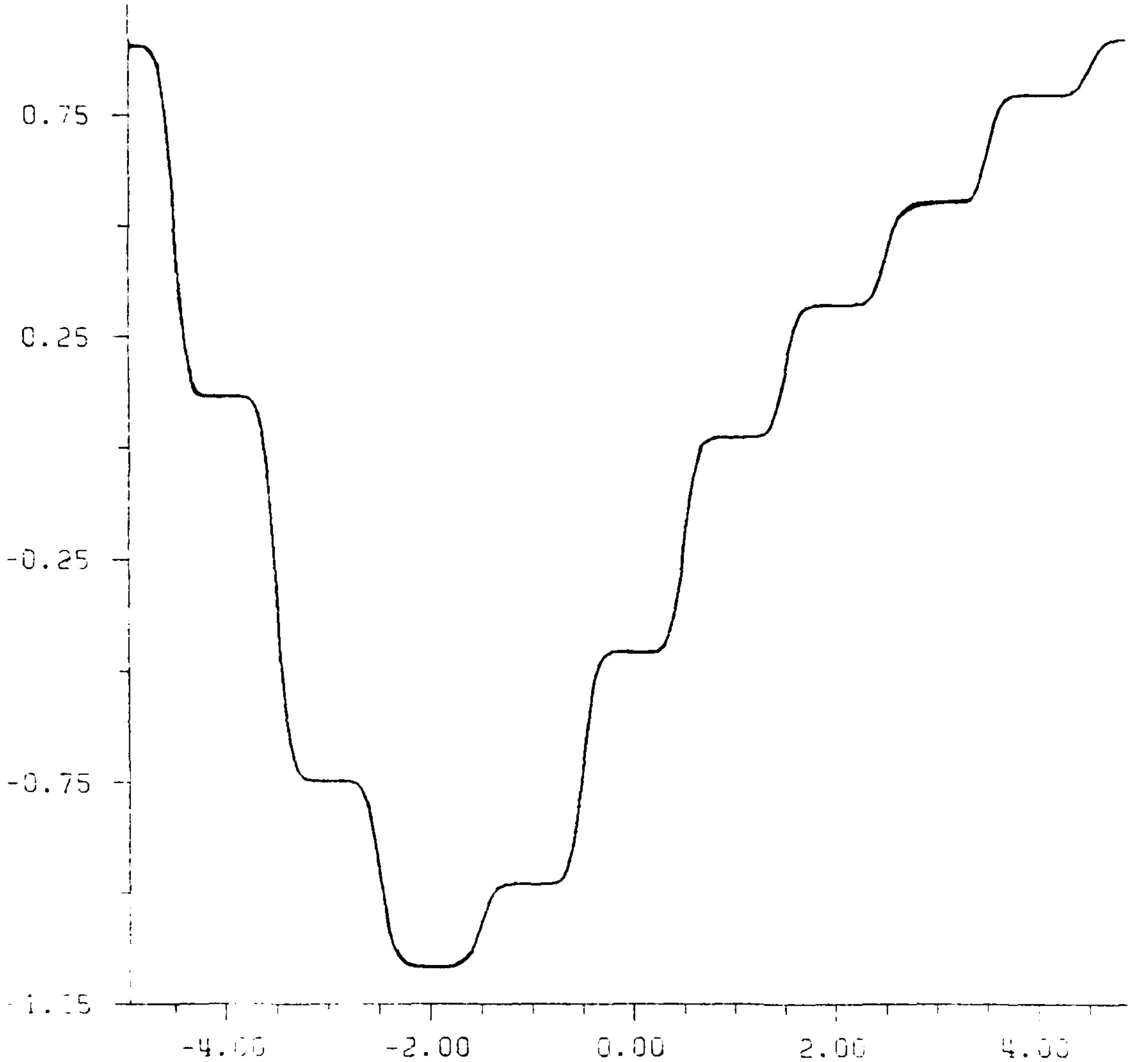

FIG. 4.1 Example of Bjerhammar interpolation (p = 3)

The other extreme is obtained if p goes to -1 : this "interpolation" function reproduces (as all the others) the data; in between the data it assumes a constant value equal to the average function value of the data; it is, therefore, a constant "function" having peaks at the data points.

A remarkable and important property of Bjerhammar interpolation is the following: the minima and maxima of the interpola-

tion function coincide with the minimal and maximal function values of the data; this follows immediately from

$$\hat{f}(x) = \frac{\sum_m f_m w_m(x)}{\sum_m w_m(x)} \geq f_{min} \frac{\sum_m w_m(x)}{\sum_m w_m(x)} = f_{min} \quad ,$$

$$\hat{f}(x) = \frac{\sum_m f_m w_m(x)}{\sum_m w_m(x)} \leq f_{max} \frac{\sum_m w_m(x)}{\sum_m w_m(x)} = f_{max} \quad ; \tag{4.10}$$

($w_m(x)$ are the positive weight functions $(x-m)^{-(p+1)}$;)

$$f_{min} \leq \hat{f}(x) \leq f_{max} \quad . \tag{4.10'}$$

Summarizing it can be said that the Bjerhammar interpolation with odd power of prediction p uses the spectrum of the corresponding spline sampling function as its sampling function; these functions show a strong tendency to approach a step function with increasing p ; even with low values of p (cf. Fig. 4.1) a step-like interpolation function is produced; therefore, it can (if at all) only be used if the function is very smooth, if it has only small slopes, and if it is well sampled. Bjerhammar and others recommend a power $p = 2.5$; I consider this value as too high and suggest to use $0.5 < p < 1.5$.

5. INTERPOLATION ERROR ESTIMATES

In chapter 3 it was shown, how least-squares sampling functions change with the correlation length; the tendency to approach the polynomial behavior corresponding to the $\sin\pi x/\pi x$ sampling function was obvious. Particularly interesting is the following question: how good does the data set represent the function. This is a pure sampling problem and the answer can be given in terms of the least-squares interpolation error estimate,

$$m_P^2 = C_{PP} - C_P^T C^{-1} C_P . \tag{5.1}$$

It has been shown that the least-squares sampling function $\psi(x)$, corresponding to the covariance (= base) function $\phi(x)$, equals

$$\psi(x-i) = \{\phi(i-j)\}^{-1}\phi(x-j) . \tag{5.2}$$

With this notation, the predicted error variance (5.1) reduces for an infinite data set at the cardinal numbers to

$$m^2(x) = \phi(0) - \sum_{j=-\infty}^{\infty} \psi(x-j)\phi(x-j) . \tag{5.1'}$$

The interpolation error has to vanish if the argument x is any integer; therefore, in this case the infinite sum should be equal to $\phi(0)$. Observing the sampling property of ψ, $\psi(j) = \delta_{0j}$, the infinite sum reduces to a single component $\phi(0)$ and

$$\begin{aligned} m^2(k) &= \phi(0) - \sum_{j=-\infty}^{\infty} \psi(k-j)\phi(k-j) \\ &= \phi(0) - \phi(0) = 0 \end{aligned}$$

follows. Due to symmetry, the maximum interpolation error occurs just in between the data $(x = \frac{1}{2} + k)$,

$$M := \max\{m(x)\} = m(\tfrac{1}{2} + k) \quad . \tag{5.3}$$

From practical considerations, we know that M depends on the correlation length h of the underlying covariance function : a small h leads to large interpolation errors, a large h to small interpolation errors. In the following, three frequently used covariance models are considered which have the common form

$$\phi(x) = \phi(0) \left[1 + \left(\frac{x}{a}\right)^2 \right]^{-\alpha} \quad . \tag{5.4}$$

Particularly well-known is the model $\alpha = 1$ which has exclusively been used by Hirvonen as early as 1962. The other two models, $\alpha = \frac{1}{2}$ and $\alpha = \frac{3}{2}$, have been discussed by Moritz (1976) because of their simple relation to their spatial spherical analogues. The correlation length h is related to the parameter a through

$$h = a\,(2^{1/\alpha} - 1)^{1/2}$$

which reduces to $h = a$ for the Hirvonen model.

The following Fig. 5.1 shows the maximum interpolation error M (eq. 5.3), depending on the correlation length h , for these three covariance models; a variance $\phi(0) = 1$ has been assumed. For comparison purposes M has also been plotted for the corresponding Gaussian covariance function.

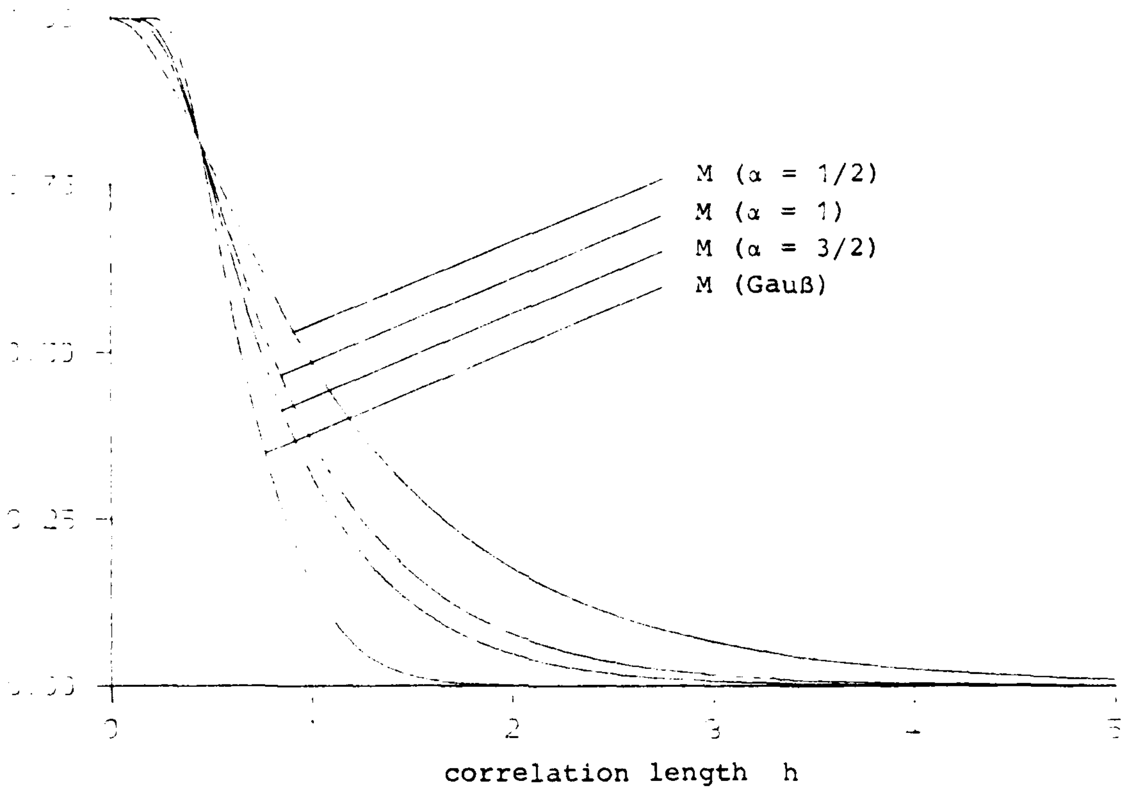

FIG. 5.1 Maximum interpolation error M

The following conclusion can be drawn from the graphs:

a) the maximum interpolation error drops fast in the range $h < 2$ and slowly in the range $h > 2$; in terms of the data density ρ ,

$$\rho = \frac{h}{\Delta} = \frac{\text{correlation length}}{\text{distance between data points}} \quad , \tag{5.5}$$

this means that M decreases slowly for $\rho > 2$. It is evident that M assumes a maximum, which equals the full variance, if $\rho = 0$ (no data) and a minimum, which is equal to zero, if

$\rho \to \infty$ (total data coverage).

b) In the range $h > 0.5$ ($\rho > 0.5$) the maximum interpolation error decreases with increasing α.

c) It is remarkable that, even with the $\alpha = 0.5$ covariance function, M is as low as 3 % of the square root of the variance for $\rho = 4$.

d) The choice of the covariance function becomes extremely critical for $\rho < 2$ because of the large differences in the interpolation error estimates. (This statement is only valid if the data are free of noise as will be shown later.)

e) The Gaussian covariance function gives by far too optimistic estimates.

A practical example may illustrate that: assume a variance of 500 $mgal^2$, a correlation length of 40 km, a reciprocal distance type covariance function ($\alpha = 1/2$), and a gravity data density of $\rho = 4$ (station distance = 10 km); on the basis of these data the maximum interpolation error is on the order of ±0.5 mgal; a station distance of 20 km leads to an estimate of ±4 mgal.

It is worth being mentioned that for a data density up to $\rho \leq 4$ the error estimate depends primarily on the very close neighborhood of the interpolation point; quite a good estimate can, therefore, be obtained by simply considering the contribution of the 2 closest data points. Equation (5.1) yields

$$M^2 \approx \phi(0) - \frac{2\phi^2(\Delta/2)}{\phi(0) - \phi(\Delta)} . \qquad (5.6)$$

The picture of maximum interpolation error behavior changes drastically if data noise is introduced. Let us first investigate how the sampling function changes if the data are superimposed by

mutually uncorrelated errors with an error variance ε ; we assume that $\varepsilon \ll \phi(0)$. For a sufficiently stable covariance matrix C of exact data, the incorporation of noise will change the corresponding inverse only little; therefore, a linearization of the inversion process can be made and the inverse covariance matrix of the data, superimposed by noise, is given by

$$(C + \varepsilon I)^{-1} \doteq C^{-1}(1 - \varepsilon C^{-1}) \quad .$$

In terms of sampling functions $\psi(x)$ and base functions $\phi(x)$, this means that, with

$$\tilde{\psi}(x-i) := \{\phi(i-j) + \varepsilon\delta_{i-j}\}^{-1}\phi(x-j) \quad , \qquad (5.7)$$

$\tilde{\psi}$ is given by

$$\tilde{\psi}(x-i) \doteq \psi(x-i) - \varepsilon\{\phi(i-j)\}^{-1}\psi(x-j) \quad . \qquad (5.7)'$$

Since ψ is a sampling function, it follows that

$$\tilde{\psi}(0) \doteq 1 - \varepsilon\phi_0^{-1}$$

where ϕ_0^{-1} denotes the diagonal element of the inverse covariance matrix. We know that ϕ_0^{-1} is a positive quantity (this follows from the property of the covariance matrix considered here); ε is als positive (a priori data error variance). Therefore,

$$\tilde{\psi}(0) < \psi(0) = 1 \quad . \qquad (5.8)$$

It can be shown that $\tilde{\psi}$ flattens out with increasing ε . For noisy data, the prediction error is given by

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$$\tilde{m}^2(x) = \phi(0) - \sum_{-\infty}^{\infty} \tilde{\psi}(x-k)\,\phi(x-k) \quad . \tag{5.9}$$

Introducing (5.7)' for $\tilde{\psi}$ we obtain

$$\tilde{m}^2(x) \doteq \phi(0) - \sum_k \psi(x-k)\,\phi(x-k) + \varepsilon \sum_k \sum_j \phi^{-1}(k-j)\,\psi(x-j)\,\phi(x-k) \;;$$

observing (5.1)' and the symmetry properties of ψ and ϕ , the prediction error based on noisy data is given by

$$\tilde{m}^2(x) \doteq m^2(x) + \varepsilon \sum_{k=-\infty}^{\infty} \psi^2(x-k) \tag{5.10}$$

ψ^2 is non-negative; therefore,

$$\tilde{m}^2(x) > m^2(x) \qquad \forall\ x \tag{5.11}$$

as to be expected. Fig. 5.2 shows $\tilde{m}(\tfrac{1}{2})$, dependent on the data density, for various covariance models. The variance was put equal to 1 , the a priori data error variance equal to 0.01 .

The graphs show very clearly that the prediction error estimation is much less sensitive with respect to the covariance function if noise is introduced (cf. Fig. 5.1). In this particular case (ε = 1 % of the variance) there is practically no gain in accuracy achievable if the data density ρ is greater than 3 . A practical example (the same as before, here, however, with non-vanishing $\varepsilon = 0.01 \cdot \mathrm{var}\{\Delta g\} = 5\ \mathrm{mgal}^2$) may illustrate that: with a gravity data density $\rho = 2$ (station distance = 20 km) a prediction error just in between the data is estimated as ± 4.4 mgal; (note that this value is only 10 % higher compared to the $\varepsilon = 0$

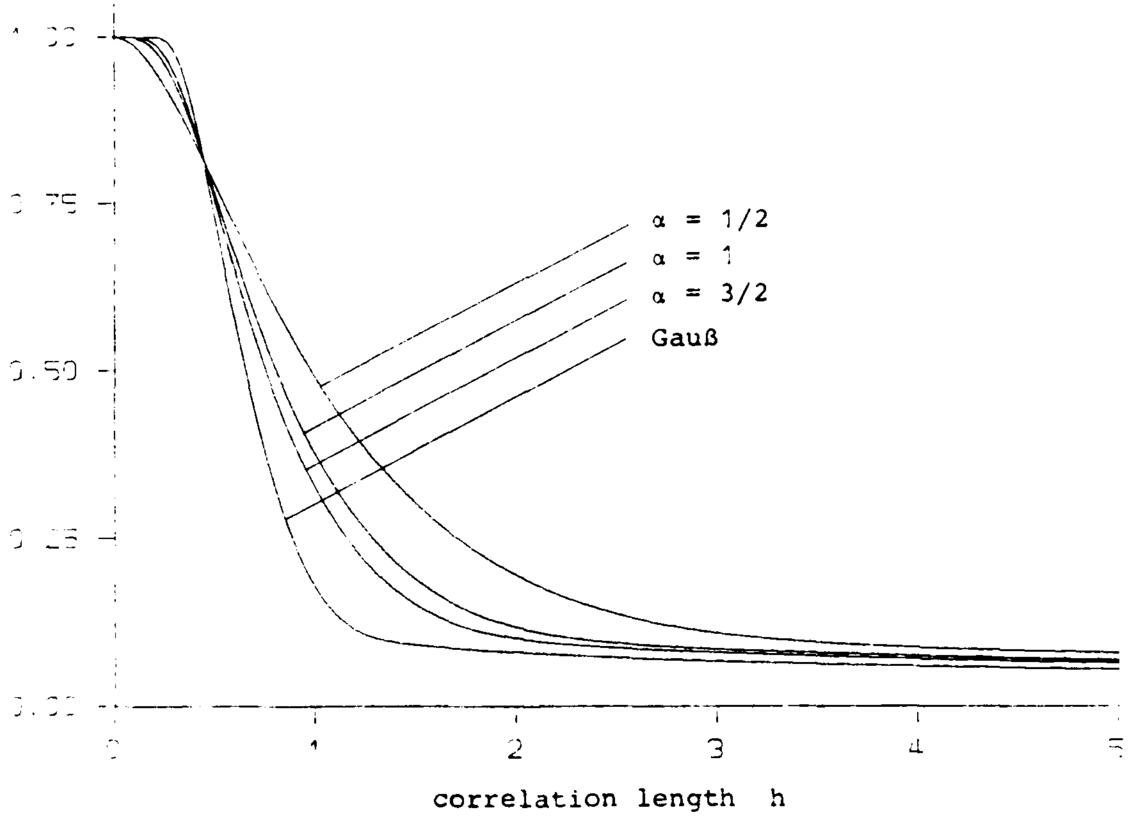

FIG. 5.2 Estimation of the prediction error $\tilde{m}(1/2)$

case - the interpolation error contributes 90 % of the figure). If the station distance is reduced to 10 km ($\rho = 4$) , the prediction error drops to ± 2 mgal and does not decrease significantly with increasing data density.

Quite a remarkable phenomenon can be observed in connection with least-squares prediction based on regularly distributed noisy homogeneous data : under certain, but certainly not unusual circumstances, the prediction error at non-observed points can be smaller than the prediction error at the data points. In the case discussed above ($\varepsilon = 0.01 \cdot \mathrm{var}\{\Delta g\}$) , this phenomenon can already be observed, if the station distance is smaller than 12 km ($\rho > 3.3$) .

The reason is the following: at the data points there is no prediction, but only filtering involved; therefore, the "prediction" error stems only from data noise. In between the data interpolation and error propagation contribute to the prediction error. For sufficiently large correlation length (or equivalently, for sufficiently high data density) the interpolation error's contribution becomes insignificant (actually, it goes to zero if $\delta \to \infty$); the prediction error stems almost exclusively from the error propagation. If the errors are not correlated, the propagated error is, indeed, smaller than the individual error (e.g. error of the mean value). This is why such a curious phenomenon can be observed in least-squares prediction.

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KEY WORDS

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Bjerhammar interpolation
Least-squares prediction
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